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Quasi-Invariant Parameterisations and Matching of Curves in Images

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Abstract. In this paper, we investigate quasi-invariance on a smooth manifold, and show that there exist *quasi-invariant parameterisations* which are not exactly invariant but approximately invariant under group transformations and do not require high order derivatives. The affine quasi-invariant parameterisation is investigated in more detail and exploited for defining general affine semi-local invariants from second order derivatives only. The new invariants are implemented and used for matching curve segments under general affine motions and extracting symmetry axes of objects with 3D bilateral symmetry.

Keywords: quasi-invariant parameterisations, semi-local invariants, integral invariants, differential invariants, curve matching, bilateral symmetry

1. Introduction

The distortions of an image curve caused by the relative motion between the observer and the scene can be described by specific *transformation groups* (Mundy and Zisserman, 1992). For example, the corresponding pair of contour curves of a surface of revolution projected on to an image center can be described by a transformation of the Euclidean group as shown in Fig. 1 (a). If a planar object has bilateral symmetry viewed under weak perspective, the corresponding contour curves of the object in an image can be described by the special affine group (Kanade and Kender, 1983; Van Gool et al., 1995a) (see Fig. 1 (b)). The corresponding contour curves of a 3D bilateral symmetry such as a butterfly are related by a transformation of the general affine group (see Fig. 1 (c)). Since these corresponding curves are equivalent objects, they have the same invariants under the specific transformation group. Thus, invariants under these transformation groups are very important for object recognition and identification (Moons et al., 1995; Mundy and Zisserman, 1992; Pauwels et al., 1995; Rothwell et al., 1995; Van Gool et al., 1992; Weiss, 1993).

Although geometric invariants have been studied extensively, existing invariants suffer from occlusion (Abu-Mostafa and Psaltis, 1984; Hu, 1962; Reiss, 1993; Taubin and Cooper, 1992), image noise (Cyganski et al., 1987; Weiss, 1988) and the requirement of point or line correspondences (Barrett and Payton, 1991; Rothwell et al., 1995; Zisserman et al., 1992). To cope with these problems, semi-local integral invariants were proposed (Bruckstein et al., 1993; Sato and Cipolla, 1996a; Sato and Cipolla, 1996b) recently. They showed that it is possible to define invariants semi-locally, by which the order of derivatives in invariants can be reduced from that of group curvatures to that of group arc-length, and hence the invariants are less sensitive to noise. As we have seen in these works, the invariant parameterisation guarantees unique identification of corresponding intervals on image curves and enables us to define semi-local integral invariants even under partial occlusions.

Although semi-local integral invariants reduce the order of derivatives required, it is known that the order of derivatives in group arc-length is still high in the general affine and projective cases (see table 1). In this paper, we introduce a *quasi-invariant parameterisation* and show how it enables us to use second order derivatives instead of fourth and fifth. The idea of quasi-invariant parameterisation is to approximate the group invariant arc-length by lower order derivatives. The new parameterisations are therefore less sensitive to noise, and are approximately invariant under a slightly restricted range of image distortions.

The concept of quasi-invariants was originally proposed by Binford (Binford and Levitt, 1993), who showed that quasi-invariants enable a reduction in the number of corresponding points required for computing algebraic invariants. For example quasi-invariants require only four points for computing planar projective invariants (Binford and Levitt, 1993), while exact planar projective invariants require five points (Mundy and Zisserman, 1992). It has also been shown that quasi-invariants exist even under the situation where the exact invariant does not exist (Binford and Levitt, 1993). In spite of its potential, the quasi-invariant has not previously been studied in detail. One reason for this is that the concept of *quasiness* is rather ambiguous and is difficult to formalise. Furthermore, the existing method is limited to the quasi-invariants based on point cor-

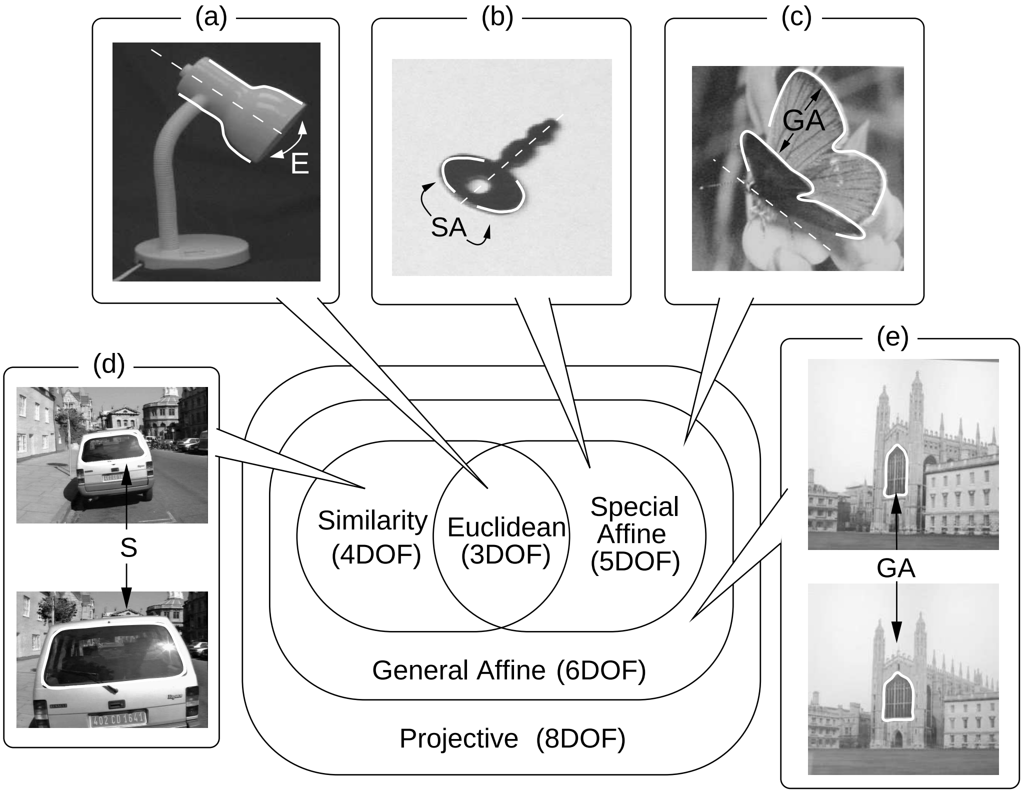
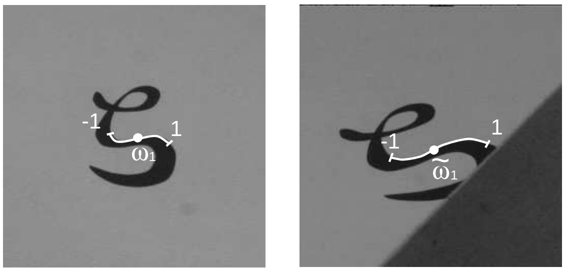


Fig. 1. Image distortion and transformation groups. A symmetric pair of contour curves (white curves) of (a) a surface of revolution, (b) planar bilateral symmetry and (c) 3D bilateral symmetry can be described by Euclidean, special affine (equi-affine) and general affine (proper affine) transformations under the weak perspective assumption. The image distortion caused by the relative motion between the observer and the scene can also be described by group transformations as shown in (d) and (e).



(a) (b)

Fig. 2. Identifying interval of integration semilocally. (a) and (b) are images of a Japanese character extracted from the first and the second viewpoints. The interval of integration in these two images can be identified uniquely from invariant arc-length, w. For example, if w_1 and $\widetilde{w}_1$ are a corresponding pair of points, then the interval [-1, 1] with respect to w corresponds to the interval [-1, 1] with respect to $\widetilde{w}$ in the second image. Even though the curve is occluded partially (second image), the semi-local integral invariants can be defined on the remaining parts of the curve.

respondences (Binford and Levitt, 1993), or the quasi-invariants under specific models (Zerroug and Nevatia, 1993; Zerroug and Nevatia, 1996).

In this paper, we investigate quasi-invariance on smooth manifolds, and show that there exists a *quasi-invariant parameterisation*, that is a parameterisation approximately invariant under group transformations. Although the approximated values are no longer exact invariants, their changes are negligible for a restricted range of transformations. Hence, the aim here is to find in parameterisations the best tradeoff between the error caused by the approximation and the error caused by image noise.

Following the motivation, we investigate a measure of invariance which describes the difference from the exact invariant under group transformations. To formalise a measure of invariance in differential formulae, we introduce the so called *prolongation* (Olver, 1986) of vector fields. We next define a quasi-invariant parameter as a function which minimises the difference from the exact invariant. A quasi invariant parameter under general affine transformations is then proposed. The proposed parameter is applied to semi-local integral invariants and exploited successfully for matching curves under general affine transformations in real image sequences.

2. Semi-Local Integral Invariants

In this section, we review semi-local integral invariants, and motivate the new parameterisation, *quasi-invariant parameterisation.*

If the invariants are too local such as differential invariants (Cyganski et al., 1987; Weiss, 1988), they suffer from noise. If the invariants are too global such as moment (integral) invariants (Abu-Mostafa and Psaltis, 1984; Hu, 1962; Lie, 1927; Reiss, 1993; Taubin and Cooper, 1992), they suffer from occlusion and the requirement of correspondences. It has been shown recently (Sato and Cipolla, 1996a; Sato and Cipolla, 1996b) that it is possible to define integral invariants semi-locally, so that they do not suffer from occlusion, image noise and the requirement of correspondences.

Consider a curve, $\mathbf{C} \in \mathbf{R}^2$, to be parameterised by t. It is also possible to parameterise the curve by invariant parameters, w, under specific transformation groups. These are called arc-length of the group. The important property of group arc-length in integral formulae is that it enables us to identify the corresponding interval of integration automatically. Consider a point, $\mathbf{C}(w_1)$, on a curve $\mathbf{C}$ to be transformed to a point, $\widetilde{\mathbf{C}}(\widetilde{w}_1)$, on a curve $\widetilde{\mathbf{C}}$ by a group transformation as shown in Fig. 2. Since $\Delta\widetilde{w} = \Delta w$, it is clear that if we take the same interval $[-\Delta w, \Delta w]$ around $\mathbf{C}(w_1)$ and $\widetilde{\mathbf{C}}(\widetilde{w}_1)$, then these two intervals correspond to each other (see Fig. 2). That is, by integrating with respect to the group arc-length, w, the corresponding interval of integration of the original and the transformed curves can be uniquely identified.

We now define semi-local integral invariants at point $\mathbf{C}(w_1)$ with interval $[-\Delta w, \Delta w]$ as follows:

$$I(w_1) = \int_{w_1-\Delta w}^{w_1+\Delta w} F dw \tag{1}$$

where, F is any invariant function under the group. The choice of F provides various kinds of semi-local integral invariants (Sato and Cipolla, 1996b). If we choose the function F carefully, the integral formula (1) can be solved analytically, and the resulting invariants have simpler forms. For example, in the affine case, if we substitute $F(w) = [\mathbf{C}_w(w), \mathbf{C}(w_1 + \Delta w) - \mathbf{C}(w_1)]$ into (1), then the integral formula is solved analytically, and the integral invariants can be described by:

$$\begin{aligned} I(w_1) \ = \ & [\mathbf{C}(w_1 + \Delta w) - \mathbf{C}(w_1), \\ & \mathbf{C}(w_1 - \Delta w) - \mathbf{C}(w_1)] \end{aligned} \quad (2)$$

where, $[\mathbf{x}_1, \mathbf{x}_2]$ denotes the determinant of a matrix which consists of two column vectors, $\mathbf{x}_1, \ \mathbf{x}_2 \ \in \ \mathbf{R}^2$. The right hand side of (2) is actually the area made by two vectors, $\mathbf{C}(w_1 + \Delta w) - \mathbf{C}(w_1)$ and $\mathbf{C}(w_1 - \Delta w) - \mathbf{C}(w_1)$. Similar results have been proposed by Bruckstein (Bruckstein et al., 1993) by a different approach. The important properties of semi-local integral invariants are as follows:

1. The limits of integration $[-\Delta w, \Delta w]$ in semi-local integral invariants are identified uniquely in original and transformed images from invariant parameterisations. Thus, we do not need to worry about the correspondence problem caused by a heuristic search of image features.
2. Even though the curve is occluded partially as shown in Fig 2 (b), the semi-local integral invariants can be defined on the remaining parts of the curve. Thus, they do not suffer from the occlusion problem unlike classical moment invariants.
3. In general, the lowest order differential invariant of a transformation group is a group curvature, and requires second, fourth, fifth and seventh order derivatives in Euclidean, special affine, general affine and projective cases (Guggenheimer, 1977; Olver et al., 1994). The semi-local integral invariants enable us to reduce the order of derivatives required from that of group curvatures to that of group arc-length. Since as shown in table 1, the order of derivatives of group arc-length is lower than that of group curvature, the semi-local integral invariants are more practical than differential invariants.

Table 1. Order of derivatives required for the group arc-length and curvature. In general derivatives more than the second order are sensitive to noise, and are not available from images. Thus, the general affine and projective arc-length as well as curvatures are not practical.

group	arc-length	curvature
Euclidean	1st	2nd
special affine	2nd	4th
general affine	4th	5th
projective	5th	7th

From table 1 (Olver et al., 1994), it is clear that the semi-local integral invariants are useful under Euclidean and special affine cases, but they still require high order derivatives in general affine and projective cases. The distortion caused by a group transformation is often not so large. For example, the distortion caused by the relative motion between the observer and the scene is restricted because of the finite speed of the camera or object motions. In such cases, parameters approximated by lower order derivatives give us a good approximation of the exact invariant parameterisation. We call such a parameterisation a *quasi-invariant parameterisation.*

In the following sections, we define the quasi-invariant parameterisation, and derive an affine quasi-invariant parameterisation.

3. Infinitesimal Quasi-Invariance

We first derive the concept of *infinitesimal quasi-invariance*; that is quasi-invariance under infinitesimal group transformations.

3.1. Vector Fields of the Group

Let G be a *Lie group*, that is a group which carries the structure of a smooth manifold in such a way that both the group operation (multiplication) and the inversion are smooth maps (Olver, 1986). Transformation groups such as rotation, Euclidean, affine and projective groups are Lie groups. Consider an image point $\mathbf{x} \in \mathbf{R}^2$ to be transformed to an image point $\widetilde{\mathbf{x}} \in \mathbf{R}^2$ by a group transformation, $h \in G$:

$$\widetilde{\mathbf{x}} = h \cdot \mathbf{x}$$

so that a function, $I(x, y)$, with respect to x and y coordinates is transformed to $\widetilde{I}(\widetilde{x}, \widetilde{y})$ by h.

Infinitesimally we can interpret this phenomenon by an action of a vector field, $\mathbf{v}$:

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \quad (3)$$

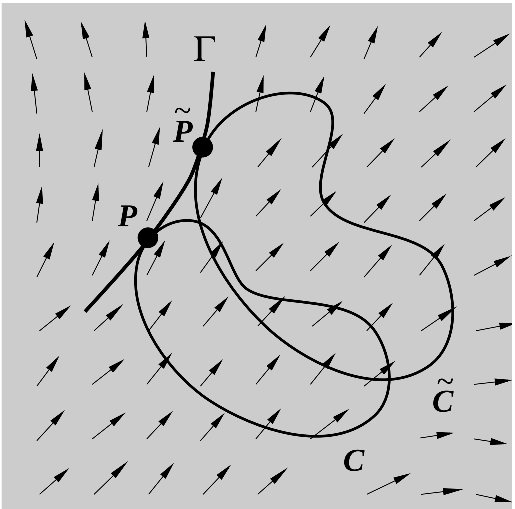


Fig. 3. The vector field, $\mathbf{v}$, and an integral curve, Γ. The curve $\mathbf{C}$ is transformed to $\widetilde{\mathbf{C}}$ by a group transformation, so that the point $\mathbf{P}$ on the curve is transformed to $\widetilde{\mathbf{P}}$. Locally the orbit of the point caused by a group transformation coincides with the integral curve, Γ, of the vector field at the point, $\mathbf{P}$.

where, ξ and η are functions of x and y, and provide various vector fields. Locally the orbit of the point, $\mathbf{x}$, caused by the transformation, h, is described by an integral curve, Γ, of the vector field, $\mathbf{v}$, passing through the point (see Fig. 3). $\mathbf{v}$ is called an *infinitesimal generator* of the group action. The uniqueness of an ordinary differential equation guarantees the existence of such a unique integral curve in the vector field.

Because of its linearity, any infinitesimal generator can be described by the summation of a finite number of independent vector fields, $\mathbf{v}_i\,(i = 1, 2, \cdots, m)$, of the group as follows:

$$\mathbf{v} = \sum_{i=1}^{m} \mathbf{v}_i \tag{4}$$

where $\mathbf{v}_i$ is the ith independent vector field:

$$\mathbf{v}_i = \xi_i \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y} \tag{5}$$

where, ξ_i and η_i are basis coefficients of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ respectively, and are functions of x and y. These independent vector fields form a finite dimensional vector space called a *Lie algebra* (Olver, 1986). Locally any transformation of the group can be described by an integral of a finite number of independent vector fields, $\mathbf{v}_i$. The vector field described in (3) acts as a differential operator of the Lie derivative.

3.2. *Exact Invariance*

We now state the condition of invariance of an arbitrary function I, which is well known in Lie Group theory.

Let $\mathbf{v}$ be an infinitesimal generator of the group transformation. A real-valued function I is invariant under group transformations, if and only if the Lie derivative of I with respect to any infinitesimal generator, $\mathbf{v}$, of the group, G, vanishes as follows (Olver, 1995):

$$\pounds_{\mathbf{v}}[\,I\,] = 0 \tag{6}$$

where $\pounds_{\mathbf{v}}[\,\cdot\,]$ denotes the Lie derivatives with respect to a vector field $\mathbf{v}$. Since I is a scalar function, the Lie derivative is the same as the directional derivative with respect to $\mathbf{v}$. Thus, the condition of invariance (6) can be rewritten as follows:

$$\mathbf{v}[\,I\,] = 0 \tag{7}$$

where $\mathbf{v}[\,\cdot\,]$ is the directional derivative with respect to $\mathbf{v}$.

3.3. *Infinitesimal Quasi-Invariance*

The idea of quasi-invariance is to approximate the exact invariant by a certain function $I(x, y)$, which is not exactly invariant but nearly invariant. If the function, I, is not exactly invariant the equation (7) no longer holds. We can however measure the difference from the exact invariant by using (7). By definition, the change in function I caused by the infinitesimal group transformation induced by a vector field, $\mathbf{v}$, is described by the Lie derivative of I as follows:

$$\begin{aligned} \delta I &= \mathbf{v}[\,I\,] \\ &= \sum_{i=1}^{m} \mathbf{v}_i[\,I\,] \end{aligned} \tag{8}$$

For measuring the invariance of a function irrespective of the choice of basis vectors, we consider an intrinsic vector field of the group, G. It is known (Sattinger and Weaver, 1986) that if the group is semi-simple (e.g. rotation group, special linear group), there exists a non-degenerate symmetric bilinear form called *Killing form*, K, of the Lie algebra as follows:

$$K(\mathbf{v}_i, \mathbf{v}_j) = tr(ad(\mathbf{v}_i)ad(\mathbf{v}_j)) \quad (i, j = 1, 2, \cdots, m) \quad (9)$$

where $ad(\mathbf{v}_i)$ denotes the adjoint representation[1] of $\mathbf{v}_i$, and tr denotes the trace. The Killing form provides the metric tensor, g_{ij}, for the algebra:

$$g_{ij} = K(\mathbf{v}_i, \mathbf{v}_j) \quad (10)$$

and the Casimir operator, C_a, defined by the metric tensor is independent of the choice of the basis vectors:

$$C_a = g^{ij}\mathbf{v}_i\mathbf{v}_j$$

where, g^{ij} is the inverse of g_{ij}. That is, the metric, g^{ij}, changes according to the choice of basis vectors, $\mathbf{v}_i$, so that C_a is an invariant. Since g^{ij} is symmetric, there exists a choice of basis vectors, $\mathbf{v}_i (i = 1, 2, \cdots, m)$, by which g^{ij} is diagonalised as follows:

$$g^{ij} = \begin{cases} \pm 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

Such vector fields, $\mathbf{v}_i (i = 1, 2, \cdots, m)$, are unique in the group, G, and thus intrinsic. By using the intrinsic vector fields in (8), we can measure the change in value of a function, I, which is intrinsic to the group, G.

For measuring the quasi invariance of a function irrespective of the magnitude of the function, we consider the change in function, δI, normalised by the original function, I. We, thus, define a measure of infinitesimal quasi invariance, Q, of a function I by the squared sum of normalised changes in function caused by the intrinsic vector fields, $\mathbf{v}_i (i = 1, 2, \cdots, m)$, as follows:

$$Q = \sum_{i=1}^{m} \left(\frac{\mathbf{v}_i[I]}{I} \right)^2 \quad (11)$$

This is a measure of how invariant the function, I, is under the group transformation. If Q is small enough, we call I a quasi-invariant under infinitesimal group transformations.

Unfortunately, if the group is not semi-simple (e.g. general affine group, general linear group), the Killing form is degenerate and we do not have such intrinsic vector fields. However, it is known that a non-semi-simple group is decomposed into a semi-simple group and a radical (Jacobson, 1962). Thus, in such cases, we choose a set of vector fields which correspond to the semi-simple group and the radical.

4. Quasi-Invariance on Smooth Manifolds

In the last section, we introduced the concept of infinitesimal quasi-invariance, which is the quasi-invariance under infinitesimal group transformations, and derived a measure for the invariance of an approximated function. Unfortunately (11) is valid only for functions which do not include derivatives. In this section, we introduce an important concept known as the *prolongation* (Olver, 1986) of vector fields, and investigate quasi-invariance on smooth manifolds, so that it enables us to define quasi-invariants with a differential formula.

4.1. Prolongation of Vector Fields

The prolongation is a method for investigating the differential world from a geometric point of view. Let a smooth curve $\mathbf{C} \in \mathbf{R}^2$ be described by an independent variable x and a dependent variable y with a smooth function f as follows:

$$y = f(x)$$

The curve, $\mathbf{C}$, is transformed to $\widetilde{\mathbf{C}}$ by a group transformation, h, induced by a vector field, $\mathbf{v}$, as shown in Fig. 4. Consider a kth order prolonged space, whose coordinates are x, y and derivatives of y with respect to x up to kth order, so that the prolonged space is $k + 2$ dimensional. The curves, $\mathbf{C}$ and $\widetilde{\mathbf{C}}$, in 2D space are prolonged and described by space curves, $\mathbf{C}^{(k)}$ and $\widetilde{\mathbf{C}}^{(k)}$, in the $k+2$ dimensional prolonged space. The prolonged vector field, $\mathbf{v}^{(k)}$, is a vector field in $k + 2$ dimension, which carries the prolonged curve, $\mathbf{C}^{(k)}$, to the prolonged curve, $\widetilde{\mathbf{C}}^{(k)}$ explicitly as shown in Fig. 4. More precisely, the kth order prolongation, $\mathbf{v}^{(k)}$, of a vector field, $\mathbf{v}$, is defined so that it transforms the kth order derivatives, $y^{(k)}$, of a function, $y = f(x)$, into the corresponding kth order derivatives, $\widetilde{y}^{(k)}$, of the transformed function $\widetilde{y} = \widetilde{f}(\widetilde{x})$ geometrically.

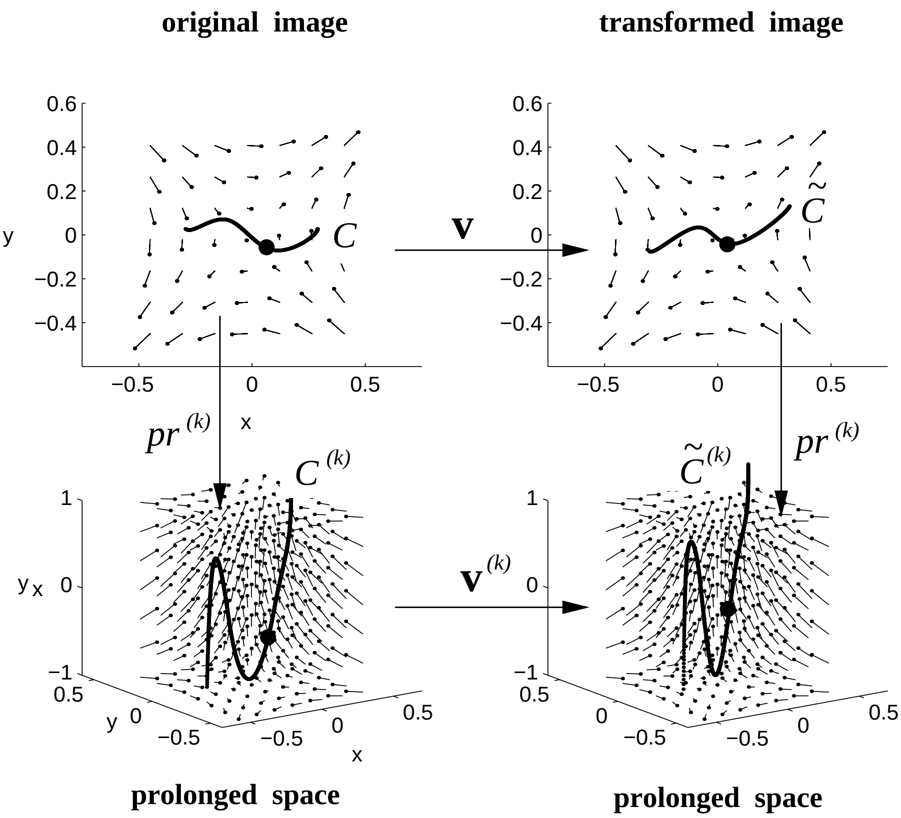


Fig. 4. Prolongation of a vector field. The kth order prolonged vector field, $\mathbf{v}^{(k)}$ transforms kth order derivatives of y into kth order derivatives of $\widetilde{y}$. That is the prolonged curve, $\mathbf{C}^{(k)}$, is transformed into the prolonged curve, $\widetilde{\mathbf{C}}^{(k)}$, by the prolonged vector field, $\mathbf{v}^{(k)}$. This enables us to investigate derivatives of functions geometrically. $pr^{(k)}$ denotes kth order prolongation. This figure illustrates the first order prolongation ($k = 1$).

Let $\mathbf{v}_i, (i = 1, .., m)$ be m independent vector fields induced by a group transformation, h. Since the prolongation is linear, the kth prolongation, $\mathbf{v}^{(k)}$, of a general vector field, $\mathbf{v}$, can be described by a sum of kth prolongations, $\mathbf{v}_i^{(k)}$, of the independent vector fields, $\mathbf{v}_i$ as follows:

$$\mathbf{v}^{(k)} = \sum_{i=1}^{m} \mathbf{v}_i^{(k)}$$

Consider a vector field (5) in 2D space again. Its first and second prolongations, $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, are computed as follows (Olver, 1986):

$$\mathbf{v}_i^{(1)} = \mathbf{v}_i + (D_x(\eta_i - \xi_i y_x) + \xi_i y_{xx})\frac{\partial}{\partial y_x} \quad (12)$$

$$\mathbf{v}_i^{(2)} = \mathbf{v}_i^{(1)} + (D_x^2(\eta_i - \xi_i y_x) + \xi_i y_{xxx})\frac{\partial}{\partial y_{xx}} \quad (13)$$

where D_x and D_x^2 denote the first and the second total derivatives with respect to x, and y_x, y_{xx}, y_{xxx} denote the first, second and the third derivatives of y with respect to x. Let $F(x, y, y^{(k)})$ be a function of x, y and derivatives of y with respect to x up to kth order, which is denoted by $y^{(k)}$. Since the prolongation describes how the derivatives are going to change under group transformations, we can compute the change in function, δF, caused by the group transformation, h, as follows:

$$\delta F = \mathbf{v}^{(k)}[F]$$

where, $\mathbf{v}^{(k)}$ is the kth order prolongation of the infinitesimal generator, $\mathbf{v}$, of a transformation h. Note that we require only the same order of prolongation as that of the function, F. Since the prolongation describes how derivatives are going to change, it is important for evaluating the quasi-invariance of a differential formula as described in the next section.

4.2. Quasi-Invariance on Smooth Manifolds

Let us consider the curve **C** in 2D space again. Suppose $I(y^{(n)})$ is a function on the curve containing the derivatives of y with respect to x up to the nth order, which we denote by $y^{(n)}$. Since the nth order prolongation, $\mathbf{v}^{(n)}$, of the vector field $\mathbf{v}$ transforms nth order derivatives, $y^{(n)}$, of the original curve to nth order derivatives, $\widetilde{y}^{(n)}$, of the transformed curve, the change in function, $\delta I_i(y^{(n)})$, caused by the infinitesimal group transformation induced by the ith independent vector field, $\mathbf{v}_i$, is described by:

$$\delta I_i(y^{(n)}) \;=\; \mathbf{v}_i^{(n)}[I(y^{(n)})] \tag{14}$$

A quasi-invariant is a function whose variation caused by group transformations is relatively small compared with its original value. We thus define a measure of invariance, Q, on smooth curve, **C**, by the normalised squared sum of $\delta I_i(y^{(n)})$ integrated along the curve, **C**, as follows:

$$Q = \int_C \sum_{i=1}^{m} \left(\frac{\mathbf{v}_i^{(n)}[I(y^{(n)})]}{I(y^{(n)})} \right)^2 dx \tag{15}$$

If $I(y^{(n)})$ is close to the exact invariant, then Q tends to zero. Thus, Q is a measure of how invariant the function, $I(y^{(n)})$, is under the group transformation.

5. Quasi-Invariant Parameterisation

In the last section, we have derived quasi-invariance on smooth manifolds. We now apply the results and investigate the quasi-invariance of parameterisation under group transformations.

A group arc-length, w, of a curve, **C**, is in general described by a group metric, g, and the independent variable, x, of the curve as follows:

$$dw = g dx$$

where, dw and dx are the differentials of w and x respectively. Suppose the metric, g, is described by the derivatives of y with respect to x up to kth order as follows:

$$g = g(y^{(k)})$$

where $y^{(k)}$ denotes the kth order prolongation of y. The change of the differential, δdw_i, caused by the ith independent vector field, $\mathbf{v}_i$, is thus derived by computing the Lie derivative of dw with respect to the kth order prolongation of $\mathbf{v}_i$:

$$\begin{aligned} \delta dw_i &= \mathbf{v}_i^{(k)}[g dx] \\ &= (\mathbf{v}_i^{(k)}[g] + g\frac{d\xi_i}{dx})dx \end{aligned} \tag{16}$$

The change in dw normalised by dw itself is described as follows:

$$\begin{aligned} \delta d\bar{w}_i &= \frac{\delta dw_i}{dw} \\ &= \frac{1}{g}\mathbf{v}_i^{(k)}[g] + \frac{d\xi_i}{dx} \end{aligned} \tag{17}$$

The measure of invariance of the parameter, w, is thus described by integrating the squared sum of $\delta d\bar{w}_i$ along the curve, **C**, as follows:

$$Q = \int_C \mathcal{L} dx \tag{18}$$

where,

$$\mathcal{L} \;=\; \sum_{i=1}^{m} (\frac{1}{g}\mathbf{v}_i^{(k)}[g] + \frac{d\xi_i}{dx})^2 \tag{19}$$

If the parameter is close to the exact invariant parameter, Q tends to zero. Although there is no exact invariant parameter unless it has enough orders of derivatives, there still exists a parameter which minimises Q and requires only lower order derivatives. We call such a parameter a quasi-invariant parameter of the group, if it minimises (18) under the linear sum of the independent vector fields of the group and keeps Q small enough in a certain range of the group transformations. To find a function, g, which minimises (18) is in fact a variational problem (Gelfand and Formin, 1963) with the Lagrangian of $\mathcal{L}$, which includes one independent variable, x, and two dependent variables, g and y (g is also dependent on y). In the next section, we derive a metric, g, which minimises the measure of invariance, Q, under general affine transformations by solving the variational problem.

6. Affine Quasi-Invariant Parameterisation

In this section, we apply quasi-invariance to derive a quasi-invariant parameterisation under general affine transformations which requires only second order derivatives and is thus less sensitive to noise than the exact invariant parameter which requires fourth order derivatives.

Suppose the quasi-invariant parameterisation, $\grave{\tau}$, under general affine transformation is of second order, so that the metric, $\grave{g}$, of the parameter, $\grave{\tau}$, is made of derivatives up to the second:

$$d\grave{\tau} = \grave{g}(y_x, y_{xx})dx \tag{20}$$

where, y_x and y_{xx} are the first and the second derivatives of y with respect to x. To find a quasi-invariant parameter is thus the same as finding a second order differential function, $\grave{g}(y_x, y_{xx})$, which minimises the quasi-invariance, Q, under general affine transformations. Since the metric, $\grave{g}$, is of second order, we require the second order prolongation of the vector fields to compute the quasi-invariance of the metric.

6.1. Prolongation of Affine Vector Fields

A two dimensional general affine transformation is described by a 2×2 invertible matrix, $A \in GL(2)$, and a translational component, $\mathbf{t} \in \mathbf{R}^2$, and transforms $\mathbf{x} \in \mathbf{R}^2$ into $\tilde{\mathbf{x}} \in \mathbf{R}^2$ as follows:

$$\tilde{\mathbf{x}} = A\mathbf{x} + \mathbf{t}$$

Since the differential form, $d\grave{\tau}$, in (20) does not include x and y components, it is invariant under translations. Thus, we here simply consider the action of $A \in \mathrm{GL}(2)$, which can be described by four independent vector fields, $\mathbf{v}_i (i = 1, \cdots, 4)$, that is the divergence, curl, and the two components of deformation (Cipolla and Blake, 1992; Kanatani, 1990; Koenderink and van Doorn, 1976):

$$\begin{aligned}
\mathbf{v}_1 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\
\mathbf{v}_2 &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \\
\mathbf{v}_3 &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \\
\mathbf{v}_4 &= y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}
\end{aligned} \tag{21}$$

Since the general linear group, GL(2), is not semi-simple, the Killing form (9) is degenerate and there is no unique choice of vector fields for the group (see section 3.3). It is however decomposed into the radical, which corresponds to the divergence, and the special linear group, SL(2), which is semi-simple and whose intrinsic vector fields coincide with $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ in (21). Thus, we use the vector fields in (21) for computing the quasi-invariance of differential forms under general affine transformations.

From (12), (13), and (21), the second prolongations of these vector fields are computed by:

$$\begin{aligned}
\mathbf{v}_1^{(2)} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - y_{xx}\frac{\partial}{\partial y_{xx}} \\
\mathbf{v}_2^{(2)} &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (1+y_x^2)\frac{\partial}{\partial y_x} + 3y_x y_{xx}\frac{\partial}{\partial y_{xx}} \\
\mathbf{v}_3^{(2)} &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2y_x\frac{\partial}{\partial y_x} - 3y_{xx}\frac{\partial}{\partial y_{xx}} \\
\mathbf{v}_4^{(2)} &= y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (1-y_x^2)\frac{\partial}{\partial y_x} - 3y_x y_{xx}\frac{\partial}{\partial y_{xx}}
\end{aligned} \tag{22}$$

These are the vector fields in four dimension, whose coordinates are x, y, y_x and y_{xx}, and the projection of these vector fields onto the x–y plane coincides with the original affine vector fields in two dimension. Since $\grave{g}dx$ is of second order, the prolonged vector fields, $\mathbf{v}_1^{(2)}$, $\mathbf{v}_2^{(2)}$, $\mathbf{v}_3^{(2)}$ and $\mathbf{v}_4^{(2)}$, describe how the parameter, $\grave{\tau}$, is going to change under general affine transformations.

6.2. Affine Quasi-Invariant Parameterisation

The measurement of the invariance, Q_a, under a general affine transformation is derived by substituting the prolonged vector fields, $\mathbf{v}_1^{(2)}$, $\mathbf{v}_2^{(2)}$, $\mathbf{v}_3^{(2)}$ and $\mathbf{v}_4^{(2)}$ of (22) into (18):

$$Q_a = \int_C \mathcal{L}(y_x, y_{xx}, \grave{g}, \grave{g}_{y_x}, \grave{g}_{y_{xx}})dx \tag{23}$$

where, $\mathcal{L}$ is a function of y_x, y_{xx}, $\grave{g}$ and its derivatives as follows:

$$\mathcal{L}(y_x, y_{xx}, \grave{g}, \grave{g}_{y_x}, \grave{g}_{y_{xx}}) = \sum_{i=1}^{4}\left(\frac{1}{\grave{g}}\mathbf{v}_i^{(2)}[\grave{g}] + \frac{d\xi_i}{dx}\right)^2 \tag{24}$$

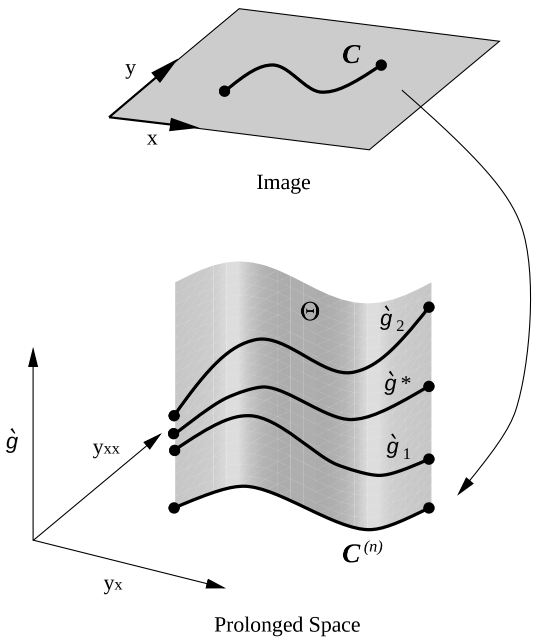


Fig. 5. Variation of $\grave{g}$. The image curve $\mathbf{C}$ is transformed to $\mathbf{C}^{(n)}$ in the prolonged space. The curve $\grave{g}$ varies only on the surface Θ defined by $\mathbf{C}^{(n)}$. What we need to do is to find a curve $\grave{g}^*$ on Θ which minimises Q_a. Thus, there is no variation in y_x and y_{xx}.

where, ξ_i denotes ξ of ith vector field, $\mathbf{v}_i$, and $\grave{g}_{y_x}$ and $\grave{g}_{y_{xx}}$ are the first derivatives of $\grave{g}$ with respect to y_x and y_{xx} respectively.

We now find a function, $\grave{g}$, which minimises, Q_a of (23). The necessary condition of Q_a to have a minimum is that its first variation, δQ_a vanishes:

$$\delta Q_a = 0 \tag{25}$$

This is a variational problem (Gelfand and Formin, 1963; Olver, 1995) of one independent variable, x, and two dependent variables, y and $\grave{g}$, and the integrand, $\mathcal{L}$, of (23) is called the Lagrangian of the variational problem. It is known that (25) holds if and only if its Euler-Lagrange vanishes as follows (Olver, 1986):

$$\mathcal{E}[\mathcal{L}] = 0$$

where, $\mathcal{E}[\,\cdot\,]$ denotes the Euler operator. Since, in our case, one of the dependent variables, $\grave{g}$, is a function of the derivatives, y_x, y_{xx}, of the other dependent variable, y, the Euler-Lagrange expression is different from the standard form of one independent and two dependent variables. We now investigate how this variational problem can be formalised.

Suppose $\grave{g}$ changes to $\grave{g}+\Delta\grave{g}$, so that $\grave{g}_{y_x}$ changes to $\grave{g}_{y_x} + \Delta\grave{g}_{y_x}$ and $\grave{g}_{y_{xx}}$ changes to $\grave{g}_{y_{xx}} + \Delta\grave{g}_{y_{xx}}$ respectively, where $\Delta\grave{g}_{y_x}$ and $\Delta\grave{g}_{y_{xx}}$ denote derivatives of $\Delta\grave{g}$ with respect to y_x and y_{xx}. Then, the first variation of the function, Q_a, caused by the change in $\grave{g}$ is described as follows:

$$\delta Q_a = \int_C \left(\frac{\partial \mathcal{L}}{\partial \grave{g}}\Delta\grave{g} + \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_x}}\Delta\grave{g}_{y_x} + \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_{xx}}}\Delta\grave{g}_{y_{xx}}\right) dx \tag{26}$$

Note that the variation occurs only on the surface Θ as shown in Fig. 5, and therefore the variation, δQ_a does not include the change in y_x and y_{xx}.

As shown in Appendix A, assuming that $\grave{g}$ has a form, ${y_{xx}}^{\alpha}(1+y_x^2)^{\beta}$ (where α and β are real values), we find that δQ_a vanishes for any curve, $y = f(x)$, if the following function, $\grave{g}$, is chosen:

$$\grave{g} = {y_{xx}}^{\frac{2}{5}}(1+y_x^2)^{-\frac{1}{10}} \tag{27}$$

We conclude that for any curve the following parameter $\grave{\tau}$ is quasi-invariant under general affine transformations:

$$d\grave{\tau} = {y_{xx}}^{\frac{2}{5}}(1+y_x^2)^{-\frac{1}{10}}dx \tag{28}$$

The quasi-invariance, Q_a, of an example curve computed by varying the power of y_{xx} and $(1+y_x^2)$ in (28) is shown in Fig. 6. We find that Q_a takes a minimum when we choose $\grave{\tau}$ shown in (28). By reformalising (28), we find that the parameter, $\grave{\tau}$, is described by the Euclidean arc-length, dv, and the Euclidean curvature, κ, as follows:

$$\begin{aligned} d\grave{\tau} &= \left(y_{xx}(1+y_x^2)^{-\frac{3}{2}}\right)^{\frac{2}{5}}(1+y_x^2)^{\frac{1}{2}}dx \\ &= \kappa^{\frac{2}{5}}dv \end{aligned} \tag{29}$$

Thus, $d\grave{\tau}$, is in fact an exact invariant under rotation, and quasi-invariant under divergence and deformation. Note, it is known that the invariant parameter under similarity transformations is κdv and that of special affine transformations is $\kappa^{\frac{1}{3}}dv$; The derived quasi-invariant parameter $\grave{\tau}$ for general affine transformations is between these two as expected. We call $\grave{\tau}$ the *affine quasi-invariant parameter (arc-length)*. Since the new parameter requires only the second order derivatives, it is expected to be less sensitive to noise than the exact invariant parameter under general affine transformations.

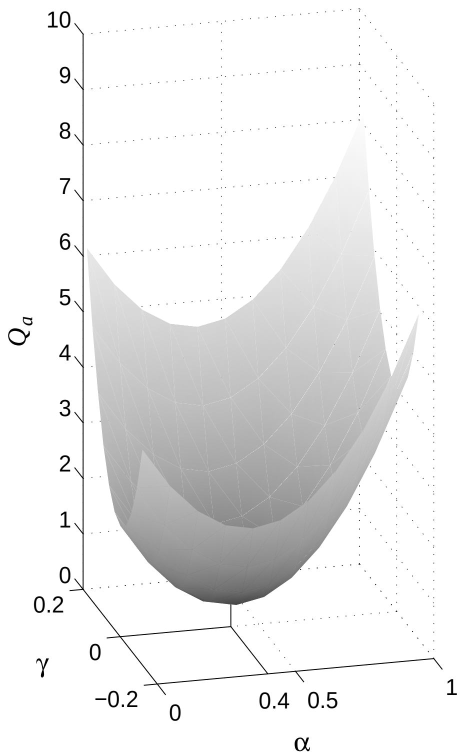


Fig. 6. Quasi-invariance of an artificial curve. The quasi-invariance, Q_a, of an example curve is computed varying the power, α and γ, in the parameter, $d\check{\tau} = {y_{xx}}^{\alpha}(1+y_x^2)^{\gamma-\frac{1}{10}}dx$. As we can see, Q_a takes a minimum at $\alpha = 0.4$ and $\gamma = 0.0$. This agrees with (28).

7. Quasi Affine Integral Invariants

In this section, we apply the extracted affine quasi-invariant parameterisation to the semi-local integral invariants described in section 2, and derive quasi integral invariants under general affine transformations.

Since $\check{\tau}$ defined in (29) is quasi-invariant, we can derive quasi integral invariants under general affine transformations by substituting $\check{\tau}$ in (29) into w in (1). If we substitute $F(\check{\tau}) = [\mathbf{C}_{\check{\tau}}(\check{\tau}), \mathbf{C}(\check{\tau}_1+\Delta\check{\tau})-\mathbf{C}(\check{\tau}_1)]$ into (1), then we have the following quasi semi-local integral invariant:

$$\begin{aligned} I(\check{\tau}_1) &= [\mathbf{C}(\check{\tau}_1+\Delta\check{\tau})-\mathbf{C}(\check{\tau}_1), \\ &\qquad \mathbf{C}(\check{\tau}_1-\Delta\check{\tau})-\mathbf{C}(\check{\tau}_1)] \qquad (30) \end{aligned}$$

which is actually the area made by two vectors, $\mathbf{C}(\check{\tau}_1+\Delta\check{\tau})-\mathbf{C}(\check{\tau}_1)$ and $\mathbf{C}(\check{\tau}_1-\Delta\check{\tau})-\mathbf{C}(\check{\tau}_1)$. The points $\mathbf{C}(\check{\tau}_1+\Delta\check{\tau})$ and $\mathbf{C}(\check{\tau}_1-\Delta\check{\tau})$ are identified by computing affine quasi-invariant arc-length, $\Delta\check{\tau} = \int d\check{\tau}$. (30) is a relative quasi-invariant of weight one under general affine transformations as follows:

$$\widetilde{I}(\widetilde{\check{\tau}}_1) = [A]I(\check{\tau}_1)$$

Since $\check{\tau}$ can be computed from just second order derivatives, the derived invariants are much less sensitive to noise than differential invariants (i.e. affine curvature). This is shown in the experiment in section 9.2.

8. Validity of Quasi-Invariant Parameterisation

Up to now we have shown that there exists a quasi-invariant parameter under general affine transformations, namely $\check{\tau}$. In this section we investigate the systematic error of the quasi-invariant parameterisation, that is the difference from the exact invariance, and show under how wide range of transformations the quasi-parameterisation is valid. As we have seen in (29), the new parameterisation is an exact invariant under rotational motion. We thus investigate the systematic error

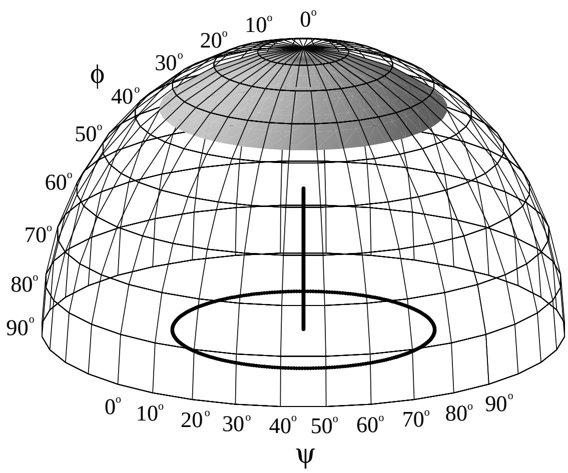


Fig. 7. Valid area of the affine quasi-invariant parameterisation. The tilt and the slant motion of a surface is represented by a point on the sphere which is pointed by the normal to the oriented disk. The motion allowed for the affine quasi-invariant parameterisation is shown by the shaded area on the sphere, which is approximately less than 35° in slant, and there is no preference in tilt angle.

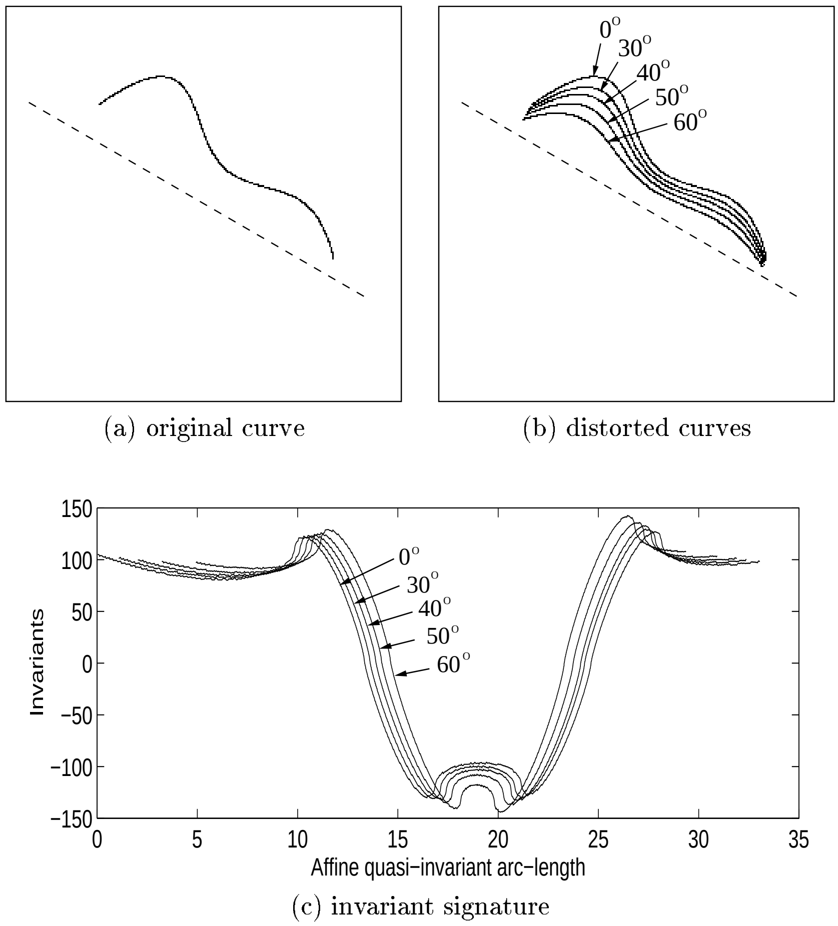


Fig. 8. Systematic error in invariant signatures. The original curve on a fronto-parallel planar surface shown in (a) is distorted by the slant motion of the surface with tilt of 60 degrees (dashed lines) and slant of 30, 40, 50, and 60 degrees as shown in (b). The distortions of the curve caused by this slant motion can be modeled by general affine transformations. (c) shows invariant signatures made of quasi semi-local integral invariants (30) extracted from the curves in (b). If the slant motion of the plane is 40 degrees or more, the invariant signature suffers from large systematic error, while if the slant is less than 40 degrees, the proposed quasi-invariants are useful.

caused by the remaining components of the affine transformation, that is the divergence and the deformation components.

From (17), the systematic error of $d\check{\tau}$ normalised by $d\check{\tau}$ itself is computed to the first order:

$$e = \sum_{i=1}^{4} a_i \left(\frac{1}{\grave{g}} \mathbf{v}_i^{(2)}[\grave{g}] + \frac{d\xi_i}{dx} \right) \qquad (31)$$

where, a_1, a_2, a_3 and a_4 are the magnitude of divergence, curl, and deformation vector fields. Substituting (22) and (27) into (31), and since the curl component of the vector field does not cause any systematic error, we have:

$$e = \frac{3}{5}a_1 - \frac{(1-y_x^2)}{5(1+y_x^2)}a_3 - \frac{2y_x}{5(1+y_x^2)}a_4 \quad (32)$$

Since both $\frac{(1-y_x^2)}{(1+y_x^2)}$ and $\frac{2y_x}{(1+y_x^2)}$ in (32) vary only from -1 to 1, we have the following inequality:

$$e \leq \frac{3}{5}|a_1| + \frac{1}{5}|a_3| + \frac{1}{5}|a_4| \qquad (33)$$

Thus, if $|a_1| \leq 0.1$, $|a_3| \leq 0.1$ and $|a_4| \leq 0.1$, then $e \leq 0.1$, and the affine quasi-invariant parameterisation is valid. Fig. 7 shows the valid area of the affine quasi-invariant arc-length (parameterisation), represented by the tilt and slant angles

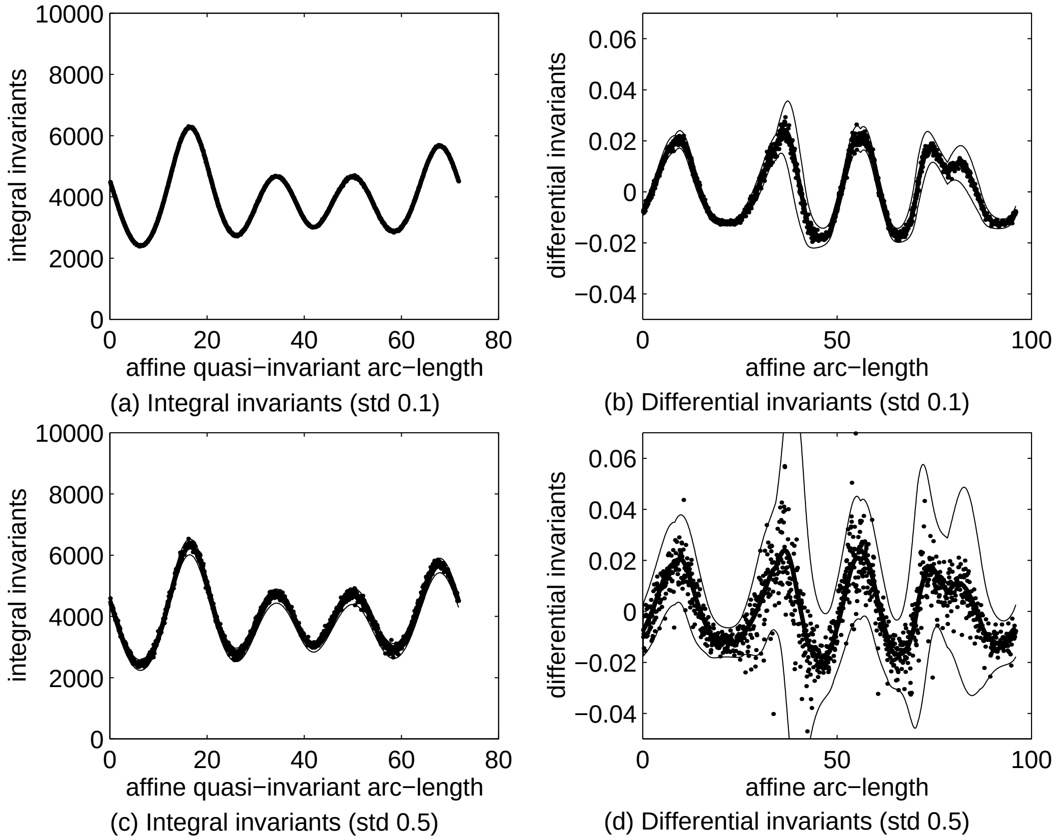


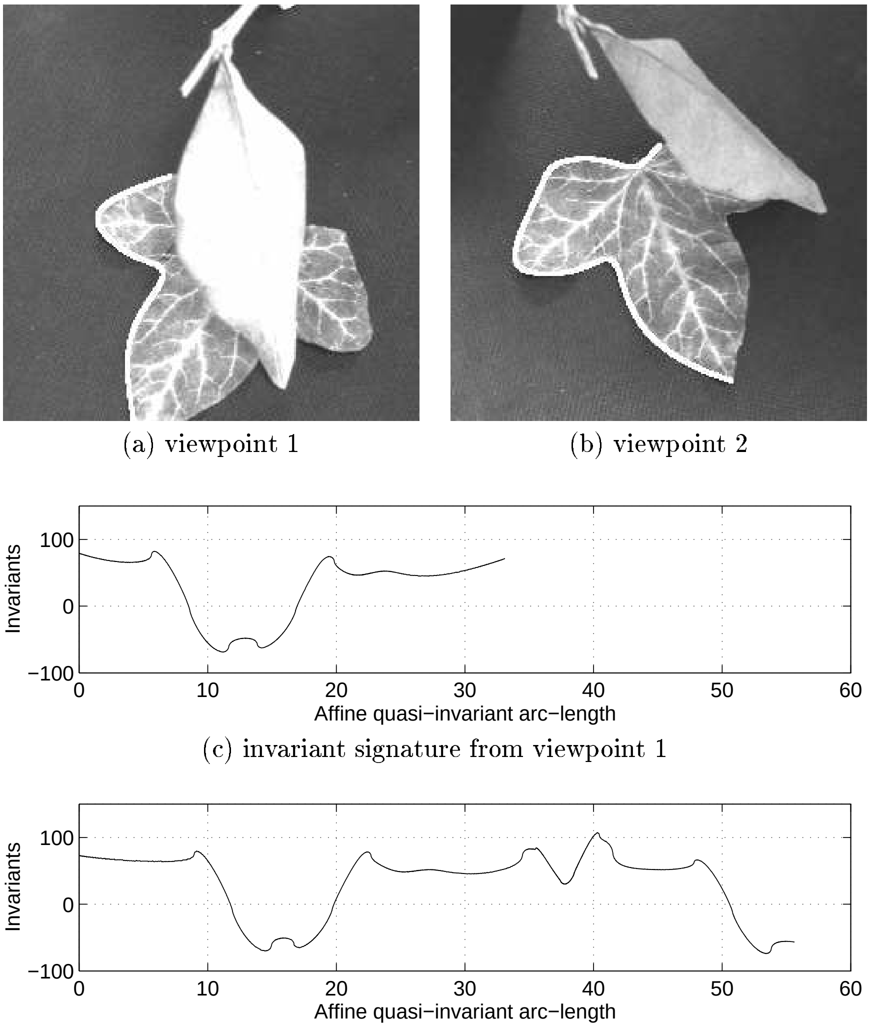
Fig. 9. Results of noise sensitivity analysis. The invariant signatures of an artificial curve are derived from the proposed invariants (semi-local invariants based on affine quasi invariant parameterisation) and the affine differential invariants (affine curvature), and are shown by thick lines in (a) and (b) respectively. The dots in (a) and (b) show signatures after adding random Gaussian noise of std 0.1 pixels, and the dots in (c) and (d) show signatures after adding random Gaussian noise of std 0.5 pixels. The thin lines show the uncertainty bounds of the signatures estimated by the linear perturbation method. The signatures from the proposed method are much more stable than those of differential invariants.

which result in the systematic error, e, smaller than 0.1. In Fig. 7, we find that if the slant motion is smaller than $35°$, the systematic error is approximately less than 0.1, and the affine quasi-invariant parameterisation is valid.

9. Experiments

9.1. *Systematic Error of Quasi Invariants*

In this section, we present the results of systematic error analysis of the quasi semi-local integral invariants, that is the semi-local integral invariants based on quasi-invariant parameterisation defined in (30), and show how large distortion is allowed for the quasi semi-local integral invariants. Fig. 8 (a) shows an image of a fronto-parallel planar surface with a curve. We slant the surface with tilt angle of 60 degrees and slant angle of 30, 40, 50 and 60 degrees as shown in Fig. 8 (b), and compute the invariant signatures of curves at each slant angle. Fig. 8 (c) shows invariant signatures of the curves computed from the quasi-invariant arc-length and the semi-local integral invariant (30). In this graph, we find that the invariant signature is distorted more under large slant motion as expected, and the proposed invariants are not valid if the slant motion is more than 40 degrees.



(a) viewpoint 1 (b) viewpoint 2

(c) invariant signature from viewpoint 1

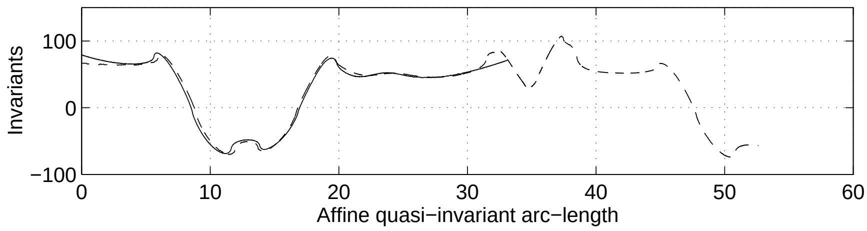
(d) invariant signature from viewpoint 2

Fig. 10. Curve matching experiment. Images of natural leaves from the first and the second viewpoint are shown in (a) and (b). The white lines in these images show extracted contour curves. The quasi-invariant arc-length and semi-local integral invariants are computed from the curves in (a) and (b), and shown in (c) and (d) respectively. In this example, we chose $\Delta\check{\tau} = 3.0$. It is clearly shown in these two signatures that the contour curve is partially occluded in (a).

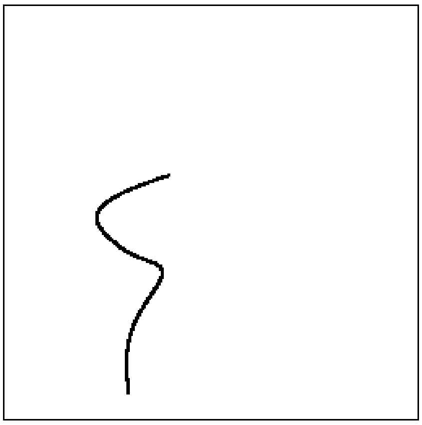
9.2. Noise Sensitivity of Quasi Invariants

We next compare the noise sensitivity of the proposed quasi semi-local integral invariants shown in (30) and the traditional differential invariants, i.e. affine curvature.

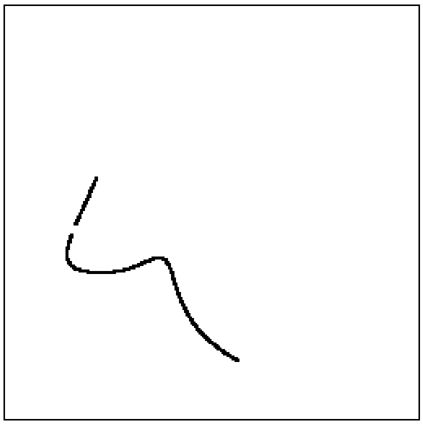
The invariant signatures of an artificial curve have been computed from the proposed quasi-invariants and the affine curvature, and are shown by solid lines in Fig. 9 (a) and (b). The dots in (a) and (b) show the invariant signatures after adding random Gaussian noise of standard deviation of 0.1 pixels to the position data of the curve, and the dots in (c) and (d) show those of standard deviation of 0.5 pixels. As we can see in these signatures, the proposed invariants are much less sensitive to noise than the differential invariants. This is simply because the proposed invariants require only second order derivatives while differential in-



(a) invariant signatures



(b) matched curves



(c) matched curves

Fig. 11. Results of curve matching experiment. The solid and dashed lines in (a) show the invariant signatures of the curves shown in Fig. 10 (a) and (b), which are shifted horizontally minimising the total difference between the two signatures. (b) and (c) show the corresponding curves extracted from the invariant signatures (a).

variants require fourth order derivatives. The thin lines show the results of noise sensitivity analysis derived by the linear perturbation method.

9.3. Curve Matching Experiments

Next we show preliminary results of curve matching experiments under relative motion between an observer and objects. The procedure of curve matching is as follows:

1. Cubic B-spline curves are fitted (Cham and Cipolla, 1996) to the Canny edge data (Canny, 1986) of each curve. This allows us to extract derivatives up to second order.
2. The quasi affine arc-length and the quasi affine semi-local integral invariants (30) with an arbitrary but constant $\Delta\hat{\tau}$ are computed at all points along a curve, and subsequently plotted as an invariant signature with quasi affine arc-length along the horizontal axis and the integral invariant along the vertical axis. The derived curve on the graph is an invariant signature up to a horizontal shift. We extract the invariant signatures of both the original and deformed curves.
3. To match curves we simply shift one invariant signature horizontally minimising the total difference between two signatures.
4. Corresponding points are derived by taking identical points on these two signatures. Even though a curve may be partially occluded or partially asymmetric, the corresponding points can be distinguished by the same procedure.

Fig. 10 (a) and (b) show the images of natural leaves taken from two different viewpoints. The white lines in these images show example contour

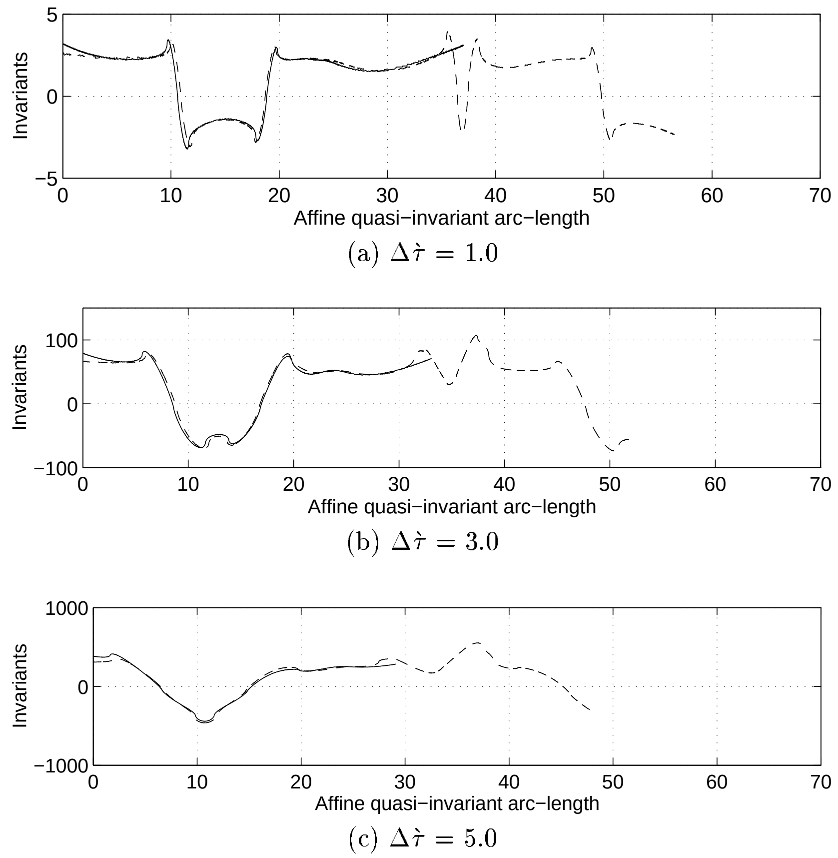


Fig. 12. Comparison of signature. The invariant signatures in (a), (b) and (c) are computed from Fig. 10 (a) and (b) by using three different $\Delta\hat{\tau}$, i.e. $\Delta\hat{\tau} = 1.0$, $\Delta\hat{\tau} = 3.0$ and $\Delta\hat{\tau} = 5.0$.

curves extracted from B-spline fitting. As we can see in these curves, because of the viewer motion, the curves are distorted and occluded partially. Since the leaf is nearly flat and the extent of the leaf is much less than the distance from the camera to the leaf, we can assume that the corresponding curves are related by a general affine transformation.

The computed invariant signatures of the original and the distorted curves are shown in Fig. 10 (c) and (d) respectively. One of these two signatures was shifted horizontally minimising the total difference between these two signatures (see Fig. 11 (a)). The corresponding points on the contour curves were extracted by taking identical points in these two signatures, and are shown in Fig. 11 (b) and (c). Note that the extracted corresponding curves are fairly accurate. In this experiment, we have chosen $\Delta\hat{\tau} = 3.0$ for computing invariant signatures. For readers' reference, we in Fig. 12 compare the invariant signatures computed from three different $\Delta\hat{\tau}$.

9.4. Extracting Symmetry Axes

We next apply the quasi integral invariants for extracting the symmetry axes of three dimensional objects. Extracting symmetry (Brady and Asada, 1984; Friedberg, 1986; Giblin and Brassett, 1985; Gross and Boult, 1994; Van Gool et al., 1995a) of objects in images is very important for recognising objects (Mohan and Nevatia, 1992; Van Gool et al., 1995b), focusing attention (Reisfeld et al., 1995) and controlling robots (Blake,

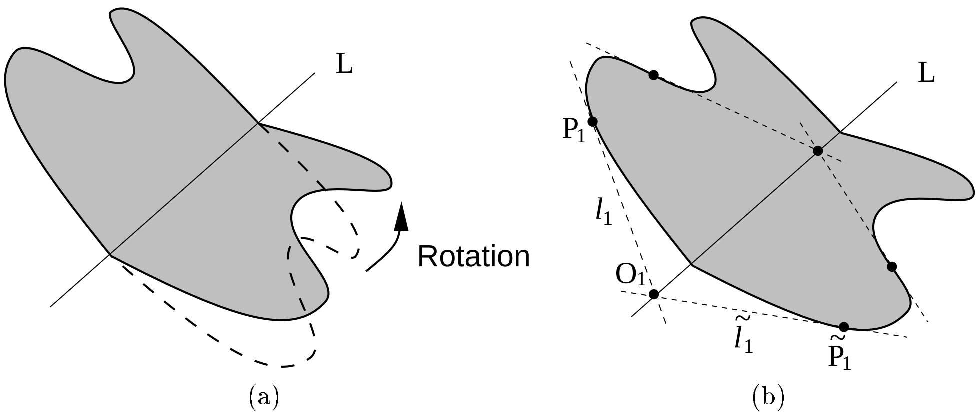


Fig. 13. Bilateral symmetry with rotation. The left and the right parts of an object with bilateral symmetry are rotated with respect to the symmetry axis, L in (a). The intersection point, O_1, of two tangent lines, l_1 and $\widetilde{l}_1$, at corresponding points, P_1 and $\widetilde{P}_1$, of a bilateral symmetry with rotation lies on the symmetry axes, L in (b). If we have N cross points, O_i $(i = 1, \cdots, N)$, the symmetry axis can be computed by fitting a line to these cross points, $O_1, O_2, \cdots, O_N$.

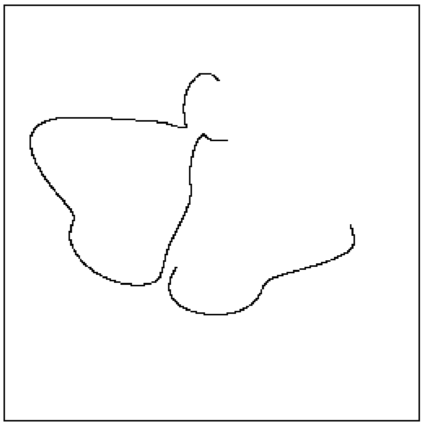
1995) reliably. It is well known that the corresponding contour curves of a planar bilateral symmetry can be described by special affine transformations (Kanade and Kender, 1983; Van Gool et al., 1995a). In this section, we consider a class of symmetry which is described by a general affine transformation.

Consider a planar object to have bilateral symmetry with an axis, L. Suppose the planar object can be separated into two planes at the axis, L, and is connected by a hinge so that two planes can rotate around this axis, L, as shown in Fig. 13 (a). The objects derived by rotating these two planes have a 3D bilateral symmetry. This class of symmetry is also common in artificial and natural objects such as butterflies and other flying insects. Since the distortion in images caused by a three dimensional motion of a planar object can be described by a general affine transformation, this class of symmetry can also be described by general affine transformations under the weak perspective assumption. Thus, the corresponding two curves of this symmetry have the same invariant signatures under general affine transformations. We must note the following properties:

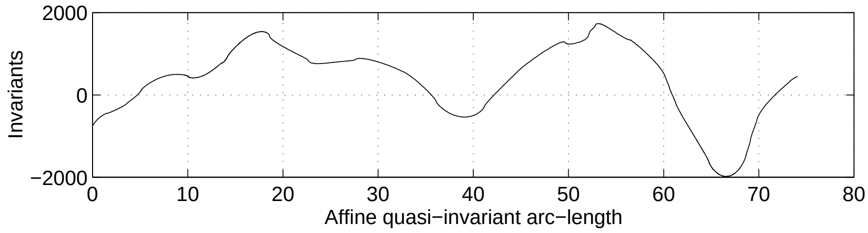
1. The skewed symmetry proposed by Kanade (Kanade, 1981; Kanade and Kender, 1983) is a special case of this class of symmetry, where the rotational angle is equal to zero and the distortion can be described by a special affine transformation with determinant of -1.
2. Unlike the skewed symmetry of planar objects, 3D bilateral symmetry takes both negative and positive determinant in its affine matrix. The positive means that the two planes are on the same side of projected symmetry axis, and the negative means that the planes are on opposite sides of the symmetry axis in the image. The extracted invariant signatures of corresponding curves of 3D bilateral symmetry are therefore either the same (i.e. positive determinant) or reflections of each other (i.e. negative determinant).
3. Unlike the skewed symmetry of planar objects, the symmetry axis of 3D bilateral symmetry is no longer on the bisecting line of corresponding symmetric curves. Instead, the cross points of the tangent lines at corresponding points on the symmetric curves lie on the symmetry axis as shown in Fig. 13 (b). Thus the symmetry axis can be extracted by computing a line which best fits to the intersection points of corresponding tangent lines.

We next show the results of extracting symmetry axes of 3D bilateral symmetry. Fig. 14 (a) shows an image of a butterfly (*Small White*) with a flower. Since the two wings of the but-

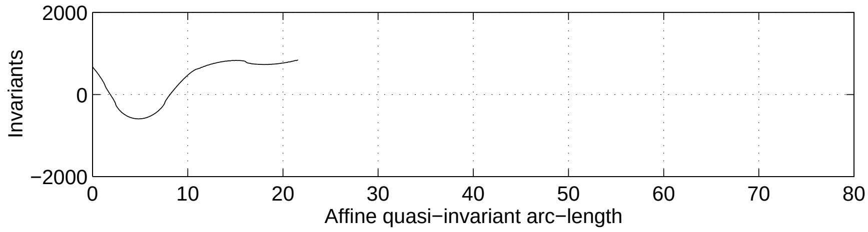
(a) original image



(b) contour curves



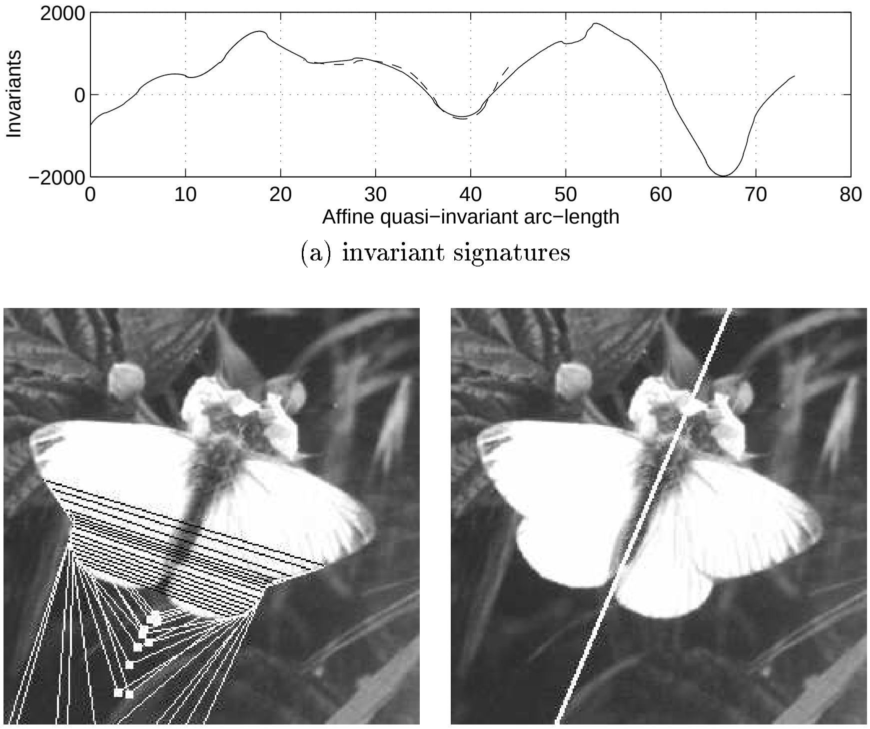
(c) invariant signature of the left wing



(d) invariant signature of the right wing

Fig. 14. Extraction of axis of bilateral symmetry with rotation. (a) shows the original image of a butterfly (*Small White*), perched on a flower. (b) shows an example of contour curves extracted by fitting B-spline curves (Cham and Cipolla, 1996) to the edge data (Canny, 1986). The invariant signatures of these curves are computed from the quasi-invariant arc-length and semi-local integral invariants. (c) and (d) are the extracted signatures of the left and the right curves in (b). In this example, we chose $\Delta\grave{\tau} = 8.0$.

terfly are not coplanar, the corresponding contour curves of the two wings are related by a general affine transformation as described above. Fig. 14 (b) shows example contour curves extracted from (a). Note that not all the points on the curves have correspondences because of the lack of edge data and the presence of spurious edges. Fig. 14 (c) and (d) shows the invariant signatures computed from the left and the right wings shown in Fig. 14 (b) respectively. (In this example, we chose $\Delta\grave{\tau} = 8.0$ for computing semi-local integral invariants.) Since the signatures are invariant up to a shift, we have simply reflected and shifted one invariant signature horizontally minimising the total difference between two signatures (see Fig. 15 (a)). As shown in these signatures, semi-local invariants based on quasi-invariant parameterisation are



(a) invariant signatures

(b) tangent lines

(c) symmetry axis

Fig. 15. Results of extracting symmetry axis of 3D bilateral symmetry. The solid and dashed lines in (a) show the invariant signatures of the curves shown in Fig. 14 (b), which are reflected and shifted horizontally minimising the total difference between two signatures. The black lines in (b) connect pairs of corresponding points extracted from the invariant signatures in (a). The white lines and the square dots show the tangent lines for the corresponding points and their cross points. The white line in (c) shows the symmetry axis of the butterfly extracted by fitting a line to the cross points.

quite accurate and stable. Corresponding points are derived by taking the identical points on these two signatures, and shown in Fig. 15 (b) by connecting the corresponding points. Tangent lines at every corresponding pair of points are computed and displayed in Fig. 15 (b) by white lines. The cross points of every pair of tangent lines are extracted and shown in Fig. 15 (b) by square dots. The symmetry axis of the butterfly is extracted by fitting a line to the cross points of tangent lines and shown in Fig. 15 (c). Although the extracted contour curves include asymmetric parts as shown in Fig. 14 (b), the computed axis of symmetry agrees with the body of the butterfly quite well. Whereas purely global methods, e.g. moment based methods (Friedberg, 1986; Gross and Boult, 1994), would not work in such cases. These results show the power and usefulness of the proposed semi-local invariants and quasi-invariant parameterisation.

10. Discussion

In this paper, we have shown that there exist quasi-invariant parameterisations which are not exactly invariant but approximately invariant under group transformations and do not require high order derivatives. The affine quasi-invariant parameterisation is derived and applied for matching of curves under the weak perspective assumption.

Although the range of transformations is limited, the proposed method is useful for many cases especially for curve matching under relative motion between a viewer and objects, since the movements of a camera and objects are, in general, limited. We now discuss the properties of the proposed parameterisation.

1. **Noise Sensitivity**
   Since quasi-invariant parameters enable us to reduce the order of derivatives required, they are much less sensitive to noise than exact invariant parameters. Thus using the quasi-invariant parameterisation is the same as finding the best tradeoff between the systematic error caused by the approximation and the error caused by the noise. The derived parameters are more feasible than traditional invariant parameters.
2. **Singularity**
   The general affine arc-length (Olver et al., 1994) suffers from a singularity problem. That is, the general affine arc-length goes to infinity at inflection points of curves, while the affine quasi-invariant parameterisation defined in (29) does not. This allows the new parameterisation to be more applicable in practice.
3. **Limitation of the Amount of Motion**
   As we have seen in section 8, the proposed quasi-invariant parameter assumes the group motion to be limited to a small amount. In the affine case, this limitation is about $a_1 \leq 0.1$, $a_3 \leq 0.1$ and $a_4 \leq 0.1$ for the divergence and the deformation components (there is no limitation on the curl component, a_2). Since, in many computer vision applications, the distortion of the image is small due to the limited speed of the relative motion between a camera and the scene or the finite distance between two cameras in a stereo system, we believe the proposed parameterisation can be exploited in many applications.

Appendix A

In this section, we derive the affine quasi-invariant arc-length, $\grave{\tau}$. As we have seen in (26), δQ_a is described as follows:

$$\delta Q_a = \int_C \left(\frac{\partial \mathcal{L}}{\partial \grave{g}} \Delta \grave{g} + \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_x}} \Delta \grave{g}_{y_x} + \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_{xx}}} \Delta \grave{g}_{y_{xx}} \right) dx \tag{A1}$$

By computing the Lie derivatives of $\grave{g}$ with respect to $\mathbf{v}_1^{(2)}$, $\mathbf{v}_2^{(2)}$, $\mathbf{v}_3^{(2)}$ and $\mathbf{v}_4^{(2)}$ in (22), we have:

$$\begin{aligned}
\mathbf{v}_1^{(2)}[\grave{g}] &= -y_{xx} \frac{\partial \grave{g}}{\partial y_{xx}} \\
\mathbf{v}_2^{(2)}[\grave{g}] &= (1 + y_x^2) \frac{\partial \grave{g}}{\partial y_x} + 3 y_x y_{xx} \frac{\partial \grave{g}}{\partial y_{xx}} \\
\mathbf{v}_3^{(2)}[\grave{g}] &= -2 y_x \frac{\partial \grave{g}}{\partial y_x} - 3 y_{xx} \frac{\partial \grave{g}}{\partial y_{xx}} \\
\mathbf{v}_4^{(2)}[\grave{g}] &= (1 - y_x^2) \frac{\partial \grave{g}}{\partial y_x} - 3 y_x y_{xx} \frac{\partial \grave{g}}{\partial y_{xx}}
\end{aligned} \tag{A2}$$

Note, that $\frac{\partial \grave{g}}{\partial x}$ and $\frac{\partial \grave{g}}{\partial y}$ components vanish. This is because $\grave{g}$ does not include x and y components, and by definition the prolonged vector fields act only on the corresponding components given explicitly (e.g. $\frac{\partial}{\partial x}$ acts only on x component and does not act on y, y_x or other components). Substituting (A2) into (24), we find that the Lagrangian, $\mathcal{L}$, is computed from:

$$\begin{aligned}
\mathcal{L} = \Bigg(& 2\grave{g}^2 (1 + y_x^2) - 4 \grave{g} y_x (1 + y_x^2) \frac{\partial \grave{g}}{\partial y_x} \\
& - 4 \grave{g} y_{xx} (2 + 3 y_x^2) \frac{\partial \grave{g}}{\partial y_{xx}} + 2 y_{xx}^2 (5 + 9 y_x^2) \left(\frac{\partial \grave{g}}{\partial y_{xx}} \right)^2 \\
& + 2 (1 + y_x^2)^2 \left(\frac{\partial \grave{g}}{\partial y_x} \right)^2 + 12 y_x y_{xx} (1 + y_x^2) \frac{\partial \grave{g}}{\partial y_x} \frac{\partial \grave{g}}{\partial y_{xx}} \Bigg) \grave{g}^{-2}
\end{aligned} \tag{A3}$$

Consider a derivative:

$$\begin{aligned}
\frac{d}{dy_x} \left(\frac{\partial \mathcal{L}}{\partial \grave{g}_{y_x}} \Delta \grave{g} \frac{dx}{dy_x} \right) &= \frac{d}{dy_x} \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_x}} \Delta \grave{g} \frac{dx}{dy_x} \\
& + \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_x}} \Delta \grave{g}_{y_x} \frac{dx}{dy_x} + \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_x}} \Delta \grave{g} \frac{d^2 x}{dy_x^2}
\end{aligned} \tag{A4}$$

By integrating both sides of (A4) with respect to dy_x, the second term of (A1) can be described by:

$$\begin{aligned}
\int_C \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_x}} \Delta \grave{g}_{y_x} dx &= \left[\frac{\partial \mathcal{L}}{\partial \grave{g}_{y_x}} \Delta \grave{g} \frac{dx}{dy_x} \right]_a^b \\
- \int_C \frac{d}{dy_x} \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_x}} \Delta \grave{g} dx & - \int_C \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_x}} \Delta \grave{g} d \frac{dx}{dy_x}
\end{aligned} \tag{A5}$$

Similarly the third term of (A1) can be described by:

$$\begin{aligned}
\int_C \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_{xx}}} \Delta \grave{g}_{y_{xx}} dx &= \left[\frac{\partial \mathcal{L}}{\partial \grave{g}_{y_{xx}}} \Delta \grave{g} \frac{dx}{dy_{xx}} \right]_a^b \\
- \int_C \frac{d}{dy_{xx}} \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_{xx}}} \Delta \grave{g} dx & - \int_C \frac{\partial \mathcal{L}}{\partial \grave{g}_{y_{xx}}} \Delta \grave{g} d \frac{dx}{dy_{xx}}
\end{aligned} \tag{A6}$$

Thus, the variation, δQ_a, is computed from (A5) and (A6) as follows:

$$\delta Q_a = E_1 + \int_C E_2 \Delta\grave{g} dx \qquad \text{(A7)}$$

where,

$$E_1 = \left[\frac{\partial\mathcal{L}}{\partial\grave{g}_{y_x}}\Delta\grave{g}\frac{dx}{dy_x}\right]_a^b + \left[\frac{\partial\mathcal{L}}{\partial\grave{g}_{y_{xx}}}\Delta\grave{g}\frac{dx}{dy_{xx}}\right]_a^b \text{(A8)}$$

$$E_2 = \frac{\partial\mathcal{L}}{\partial\grave{g}} - \frac{d}{dy_x}\frac{\partial\mathcal{L}}{\partial\grave{g}_{y_x}} - \frac{\partial\mathcal{L}}{\partial\grave{g}_{y_x}}\frac{d}{dx}\frac{dx}{dy_x} - \frac{d}{dy_{xx}}\frac{\partial\mathcal{L}}{\partial\grave{g}_{y_{xx}}} - \frac{\partial\mathcal{L}}{\partial\grave{g}_{y_{xx}}}\frac{d}{dx}\frac{dx}{dy_{xx}} \quad \text{(A9)}$$

where, a and b are the limit of integration specified by the curve, $\mathbf{C}$. From (A3), the derivatives of $\mathcal{L}$ in (A8) and (A9) can be computed by:

$$\frac{\partial\mathcal{L}}{\partial\grave{g}} = \Bigg(-4(1+y_x^2)^2\left(\frac{\partial\grave{g}}{\partial y_x}\right)^2 - 4y_{xx}^2(5+9y_x^2)\left(\frac{\partial\grave{g}}{\partial y_{xx}}\right)^2 - 24y_xy_{xx}(1+y_x^2)\frac{\partial\grave{g}}{\partial y_x}\frac{\partial\grave{g}}{\partial y_{xx}} + 4y_x(1+y_x^2)\grave{g}\frac{\partial\grave{g}}{\partial y_x} + 4y_{xx}(2+3y_x^2)\grave{g}\frac{\partial\grave{g}}{\partial y_{xx}}\Bigg)\grave{g}^{-3} \qquad \text{(A10)}$$

$$\frac{\partial\mathcal{L}}{\partial\grave{g}_{y_x}} = \left(-4y_x(1+y_x^2)\grave{g} + 12y_xy_{xx}(1+y_x^2)\frac{\partial\grave{g}}{\partial y_{xx}} + 4(1+y_x^2)^2\frac{\partial\grave{g}}{\partial y_x}\right)\grave{g}^{-2} \qquad \text{(A11)}$$

$$\frac{\partial\mathcal{L}}{\partial\grave{g}_{y_{xx}}} = \left(4y_{xx}^2(5+9y_x^2)\frac{\partial\grave{g}}{\partial y_{xx}} - 4y_{xx}(2+3y_x^2)\grave{g} + 12y_xy_{xx}(1+y_x^2)\frac{\partial\grave{g}}{\partial y_x}\right)\grave{g}^{-2} \qquad \text{(A12)}$$

Since $\Delta\grave{g}$ in (A7) must be able to take any value, δQ_a vanishes if and only if:

$$E_1 = 0 \quad \textit{and} \quad E_2 = 0 \qquad \text{(A13)}$$

The question is what sort of function, $\grave{g}$, makes the condition (A13) hold. Here, we assume that $\grave{g}$ takes the following form:

$$\grave{g} = y_{xx}^{\alpha}(1+y_x^2)^{\beta} \qquad \text{(A14)}$$

and investigate the unknown parameters α and β for (A13) to hold for arbitrary curves. Substituting (A14) into (A10), (A11), and (A12), we have:

$$\frac{\partial\mathcal{L}}{\partial\grave{g}} = \grave{g}^{-1}(4\alpha(2-5\alpha) - 4y_x^2(3\alpha+2\beta-1)(3\alpha+2\beta)) \qquad \text{(A15)}$$

$$\frac{\partial\mathcal{L}}{\partial\grave{g}_{y_x}} = 4\grave{g}^{-1}y_x(1+y_x^2)(3\alpha+2\beta-1) \qquad \text{(A16)}$$

$$\frac{\partial\mathcal{L}}{\partial\grave{g}_{y_{xx}}} = 4\grave{g}^{-1}y_{xx}((5\alpha-2)+3y_x^2(3\alpha+2\beta-1)) \qquad \text{(A17)}$$

Since y_x and y_{xx} in (A15), (A16) and (A17) take arbitrary values, the condition (A13) holds for arbitrary curves if:

$$5\alpha - 2 = 0 \quad \textit{and} \quad 3\alpha + 2\beta - 1 = 0 \qquad \text{(A18)}$$

Thus, α and β must be:

$$\alpha = \frac{2}{5} \qquad \text{(A19)}$$

$$\beta = -\frac{1}{10} \qquad \text{(A20)}$$

Substituting (A19) and (A20) into (A14), we find that the following form for $\grave{g}$ gives the extremal to Q_a:

$$\grave{g} = y_{xx}^{\frac{2}{5}}(1+y_x^2)^{-\frac{1}{10}}$$

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Notes

1. The adjoint representation, $ad(\mathbf{v}_i)$, provides a $m \times m$ matrix representation of the algebra, whose (j,k) component is described by a structure constant C_{ik}^j (Sattinger and Weaver, 1986).

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