Surface geometry from cusps of apparent contours

Roberto Cipolla Gordon Fletcher Peter Giblin

Department of Engineering
University of Cambridge
Cambridge CB2 1PZ, U.K.

Department of Pure Mathematics
University of Liverpool
Liverpool L69 3BX, U.K.

Abstract

It is known that the deformations of the apparent contours of a surface under perspective projection and viewer motion enable the recovery of the geometry of the surface, for example by utilising the epipolar parametrization. *These methods break down with apparent contours that are singular i.e. with* cusps. *In this paper we study this situation in detail and show how, nevertheless, the surface geometry (including the Gauss curvature and mean curvature of the surface) can be recovered by* following the cusps. *Indeed the formulae are much simpler in this case and require lower spatio-temporal derivatives than in the general case of nonsingular apparent contours. We give a simulated example, and also show that following cusps does not by itself provide us with information on ego-motion.*

1 Introduction

For smooth curved surfaces an important image feature is the profile or apparent contour. This is the projection of the locus of points on the surface which separates the visible and occluded parts. Under perspective projection this locus – the critical set or contour generator, Σ – can be constructed as the set of points on the surface where rays through the projection centre $\mathbf{c}$ are tangent to the surface. Each viewpoint will generate a different contour generator with the contour generators 'slipping' over the visible surface under viewer motion. If we want to associate a contour generator with a particular projection centre $\mathbf{c}(t)$ at time t, then we can write it as $\Sigma(t)$.

The contour generators give rise to visible 'apparent contours' or 'profiles' in the image. Giblin and Weiss [8] showed how, under orthogonal projection and with a special class of motions, these apparent contours determine the surface, and hence all its geometry. Cipolla and Blake [4] generalised this to perspective projection and arbitrary, but known, motion. In order to simplify the analysis, they used the *epipolar* parametrization: the correspondence between points of apparent contours on successive snapshots is set up by matching along epipolar lines. The parametrization is especially suited to the recovery of surface geometry by an active explorer making deliberate viewer motions around an object of interest and it has been successfully implemented in various systems [4, 14].

There are however several cases in which this parametrization is degenerate and so can not be used to recover the local surface geometry. One case of degeneracy occurs at a point of a contour generator where the ray through the projection centre is tangent not only to the surface but also to the contour generator. Close to such a point, the epipolar parametrization becomes *ill-conditioned*, and it is impossible to use contour generators and epipolar curves as a local coordinate grid on the surface. The geometrical condition for this degeneracy to occur is that we are viewing a hyperbolic surface patch along an asymptotic direction [12, pp.422,437]. For a transparent surface the effect on the apparent contour is to generate a *cusp*. For opaque surfaces, however, only one branch of the cusp is visible and the contour ends abruptly [10, 11], [12, p.422]. We call such a surface point a *cusp generator point* and the corresponding image point simply a *cusp point*. Under viewer motion the locus of the cusp generator points on the surface defines the *cusp generator curve*, while in the image the locus of cusp points defines the *cusp locus*.

Cusp points are extremely visible in the images of transparent objects and in X-ray imaging. Inspection of machine components such as turbine blades and biological organs increasingly exploits 3D reconstruction from X-ray images [13]. The corresponding contour endings for opaque objects are more difficult to detect, but Figure 1 shows real images obtained from a sculpture by Henry Moore in the Yorkshire Sculpture Park in England. A smooth profile (top) deforms through a 'swallowtail transition' into two cusps. In the lower two pictures the visible contour ending has been marked; the other branch and cusp is occluded by the opaque surface.

An entirely different degeneracy of the epipolar paramterization occurs at 'frontier points'. See [1, 6, 9].

In this paper we show that, despite the failure of the epipolar parametrization along the cusp generator curve, the special geometry of cusps can be used to advantage to produce an alternative parametrization using the cusp generator curve itself. This leads to a *simplified* formula for Gauss curvature, involving only first-order temporal derivatives in place of the second-order temporal and spatial derivatives required in the non-cusp case [4, §4]. Giblin and Soares [7] presented a first attempt to relate local surface geometry (Gauss

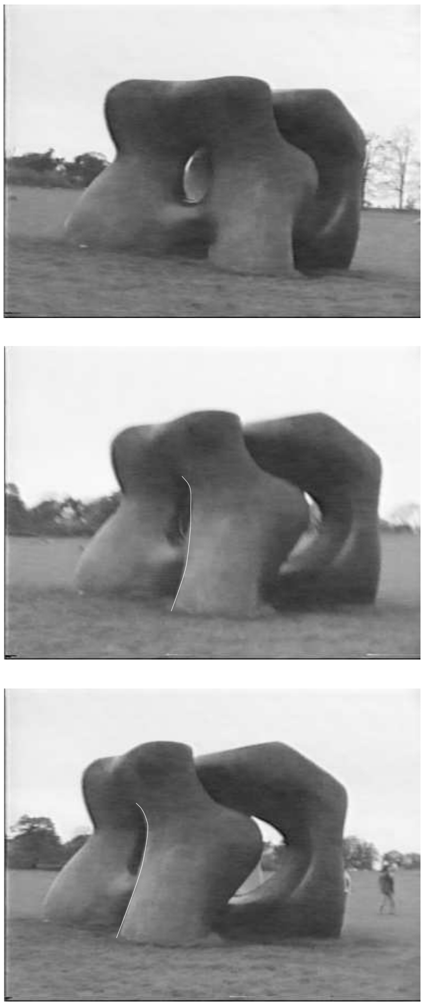

Figure 1: Sequence of real images with the lower two showing a moving contour ending

and mean curvatures and principal directions) to the image motion of cusps under orthographic projection and planar viewer motion. We extend this here to arbitrary nonplanar, curvilinear viewer motion under perspective projection. We give a simulated example, where it is possible to calculate the true Gauss curvature and compare with that measured from the cusp locus in the image. We also investigate the problems and ambiguities in attempting to recover egomotion from the image motion of cusp points.

2 The cusp generator and locus of cusps

As our camera centre $\mathbf{c}(t)$ moves in space, the contour generator Σ on the surface M and the apparent contour in the image sphere will change. In this paper, we are concerned with the case where the apparent contours all have cusps, which means that, for each t, some viewline from $\mathbf{c}(t)$ to the surface is not merely tangent at the corresponding point $\mathbf{r} \in M$ but is *asymptotic* there.

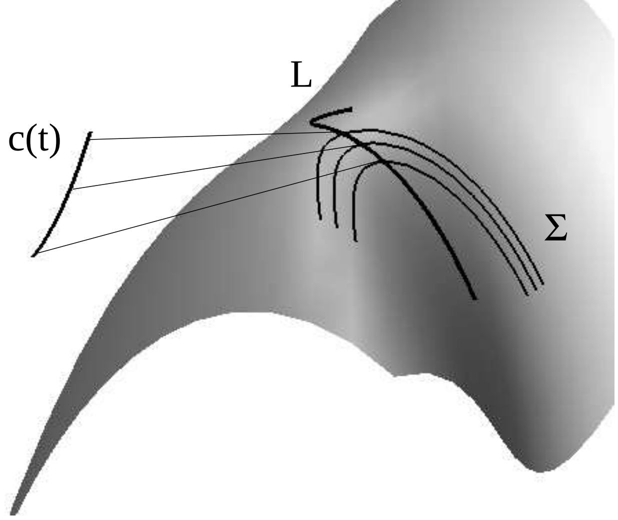

Figure 2: The case considered in this paper: the cusp generator curve L is nonsingular and transverse to the contour generators Σ. Also the cusp locus in the image sphere (not shown) is nonsingular.

Each t then gives rise to one or more points $\mathbf{r}$ of M where an asymptotic ray passes through $\mathbf{c}(t)$; these points $\mathbf{r}$ make up what we call the *cusp generator curve* L on M. In the image sphere there is a corresponding locus of points (*unit* vectors) $\mathbf{p}$ which we call the *locus of cusps* C. As in [4] we can also rotate the camera relative to the world frame by a rotation $R(t)$ depending on t. Then we have new 'rotated' coordinates $\mathbf{q}$ where $\mathbf{p} = R\mathbf{q}$ (in [4] $\mathbf{p}$, $\mathbf{q}$ appear as $\mathbf{Q}$, $\tilde{\mathbf{Q}}$ respectively).

In what follows we consider only a generic motion $\mathbf{c}$. The nature of the two curves L, C depends on the disposition of $\mathbf{c}$ relative to certain surfaces in space, namely the ruled (developable) surface of 'cylinder axes' ([12, p.298]) and the ruled surface of 'flecnodal rays' ([12, p.282]). When the motion $\mathbf{c}$ crosses one of these surfaces the apparent contours undergo 'lips/beaks' or 'swallowtail' transitions [12, p.458]. For the remainder of this paper we assume that our motion avoids these two surfaces. A typical picture of contour generators, cusp generator curve L and motion $\mathbf{c}$ is shown in Figure 2. A crucial feature is that the cusp generator curve L is smooth and transverse (non-tangential) to the smooth contour generators Σ on the surface M. Also the locus of cusps C in the image is smooth.

3 Using the cusp locus to obtain geometric information on the surface

In this section we show that by 'following cusps' we can obtain the Gauss curvature of the surface M *using only first order derivatives* of the motion and cusp locus. This is in sharp contrast with the general situation [4] where we need to use second-order derivatives both in time and space. We also compare the motion of the cusp with the motion of a 'feature' attached to the surface and show that their relative speed gives us a constraint on the motion. Note that L, being transverse to the (smooth) contour generators Σ, is parametrized by t, that is, L can be written as a set of points $\mathbf{r}(t)$ on M. (When $\mathbf{c}$ crosses the special surfaces mentioned in §2 above, this fails.)

3.1 Parametrization using the cusp generator curve L

There is an important consequence of the transversality of L and the contour generators $\Sigma(t)$ on the surface M. Namely, M can be parametrized (locally) as $\mathbf{r}(t,u)$ where t is time and u is another parameter, in such a way that $\mathbf{r}(t,0)$ is the cusp generator curve L. This parametrization can be regarded as *replacing* the epipolar parametrization when the latter becomes degenerate, as it does along L. The image consists of points (unit vectors) $\mathbf{p}(t,u)$, where as usual

$$\mathbf{r}(t,u) = \mathbf{c}(t) + \lambda \mathbf{p}(t,u).$$

For fixed t $\mathbf{p}(t,u)$ gives the apparent contour and for $u=0$ and a given t it gives the cusp on that apparent contour. In this situation $\mathbf{p}_u(t,0)=0$ for every t, since the cusp is a singular point of the parametrization of the apparent contour.

The normal to M is parallel to the *limiting normal* to the apparent contour at $\mathbf{p}(t,u)$ as $u \to 0$, that is, the limit as a apparent contour point approaches the cusp. Thus the surface normal $\mathbf{n}$ remains detectable from the image; in fact

$$\mathbf{n} \text{ is parallel to } \mathbf{p} \wedge \mathbf{p}_{uu} \text{ for ordinary cusps.} \tag{1}$$

See the Appendix. The depth formula [4, Eqn.(32)]

$$\lambda = -\mathbf{c}_t.\mathbf{n}/\mathbf{p}_t.\mathbf{n} \tag{2}$$

remains unchanged for this situation, but the formula for Gauss curvature K [*loc. cit.*, §4.3.4], is based on Koenderink's famous relationship [12, p.433] $K=\kappa^p\kappa^t/\lambda$, where κ^p is the (geodesic) curvature of the apparent contour in the image sphere and κ^t is the 'transverse' curvature of M in the direction of the viewline. In our situation this takes the form $\infty \times 0$ and so is invalid.

3.2 New formulae for depth and curvature

In what follows we shall parametrize the cusp locus L on M by t, writing it as simply $\mathbf{r}(t)$, which corresponds to $\mathbf{r}(t,0)$ in §3.1 above. In fact, our main result here concerns *any* curve $\mathbf{r}(t)$ on M which is parametrized by t; the main application is certainly to the case where, as above, this is the cusp generator curve L. The curve on M has an image $\mathbf{p}(t)$ (and a corresponding image $\mathbf{q}(t)$ in rotated coordinates). We suppose that, at $t=0$, the point $\mathbf{p}(0)$ is a singular apparent contour point (a cusp), i.e. that $\mathbf{r}(0) \in L$. Thus we assume that $\mathbf{r}(0)$ is on L, but, for deducing the results below, it does not matter whether $\mathbf{r}(t)$ continues to be a point of L.

We write λ for the depth, that is the distance from $\mathbf{c}(0)$ to $\mathbf{r}(0), K$ for Gauss curvature at $\mathbf{r}(0)$, $\mathbf{n}$ for the surface normal there, which is parallel to the (limiting) normal to the apparent contour at the cusp, and $\mathbf{t} = \mathbf{n} \wedge \mathbf{p}$, which is parallel to the cuspidal tangent. A sketch of the proof of the following proposition is given in the Appendix. (See also [5].)

Proposition 3.1 *In the above setup,*

$$[\mathbf{p},\mathbf{c}_t,\mathbf{p}_t]^2 = -\frac{(\mathbf{c}_t.\mathbf{n})^4}{\lambda^4 K}, \tag{3}$$

$$K = -\frac{(\mathbf{p}_t.\mathbf{n})^4}{[\mathbf{p},\mathbf{c}_t,\mathbf{p}_t]^2}, \tag{4}$$

$$\mathbf{p}_t.\mathbf{t} = \epsilon\frac{\mathbf{c}_t.\mathbf{n}}{\lambda^2\sqrt{-K}} - \frac{\mathbf{c}_t.\mathbf{t}}{\lambda}, \tag{5}$$

$$\mathbf{p}_t.\mathbf{n} = -\frac{\mathbf{c}_t.\mathbf{n}}{\lambda}, \tag{6}$$

where all quantities are evaluated at $t=0$ and ϵ is ± 1, given by the sign of the triple scalar product $[\mathbf{p},\mathbf{c}_t,\mathbf{p}_t]$. (Note in passing that with the epipolar parametrization $\mathbf{p}(t,u)$, $\mathbf{p}$, $\mathbf{p}_t$ and $\mathbf{c}_t$ are coplanar, so this triple scalar product is zero.)

Notes on the Proposition

1. The remarkable thing about the formula for K is that it involves only *first* derivatives in time, and no derivatives in space. This is in contrast with the formula in [4, §4], where second derivatives in both space and time are required.

2. There is a similar formula [5] for the mean curvature H. In fact, the whole second fundamental form can be derived, knowing K, H and an asymptotic direction (along the viewline $\mathbf{r}-\mathbf{c}$).

3. If $\mathbf{p}(t)$ is the image of a *fixed* point of M (i.e. $\mathbf{r} = \mathbf{c}(t) + \lambda(t)\mathbf{p}(t)$ for a constant $\mathbf{r}$), then the image velocity is

$$\mathbf{p}_t = \frac{(\mathbf{c}_t \wedge \mathbf{p}) \wedge \mathbf{p}}{\lambda} = \frac{\mathbf{p}(\mathbf{c}_t.\mathbf{p}) - \mathbf{c}_t}{\lambda}.$$

   Let us take the scalar product with the normal $\mathbf{n}$ and the tangent $\mathbf{t}$, both of which are perpendicular to $\mathbf{p}$, and compare with (5), (6). We deduce that the image of a fixed point and the image of a surface curve (for example, the cusp generator curve L) have relative velocity of magnitude

$$v = \frac{\mathbf{c}_t.\mathbf{n}}{\lambda^2\sqrt{-K}}$$

   along the cuspidal tangent. This should be compared with the calculations in [4, §5], where it is the *derivative* of parallax which has a nonzero value. Note that this moving of the image of the cusp away from the epipolar direction followed by the fixed feature gives a way of distinguishing cusps from fixed features. Compare [3].

4 An example and a simulated experiment

4.1 A general calculation

When the surface M is given by an equation $z = h(x,y)$, and the camera motion is given by a vector-valued function $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$, then it is easy to write down the conditions which must be satisfied by the cusp generator curve $\mathbf{r}(t) = (x(t), y(t), h(x(t), y(t)))$ on M. There are two conditions, both of which are obtained in the same way, as follows. Consider, for a fixed t, the line joining $\mathbf{r}(t)$ to $\mathbf{c}(t)$. This line consists of points (omitting the variable t)

$$\mathbf{r}+\mu(\mathbf{r}-\mathbf{c}) = (x+\mu(x-c_1), y+\mu(y-c_2), z+\mu(z-c_3)),$$

where μ is an arbitrary real number, taking the value 0 at $\mathbf{r}(t)$. This line meets the surface where

$$z + \mu(z - c_3) = h\left(x + \mu(x - c_1), y + \mu(y - c_2)\right). \quad (7)$$

This equation for μ naturally has $\mu = 0$ as a solution; we want to impose the conditions that $\mu = 0$ is a *triple* solution, since this means that the line has contact 3 with the surface, i.e., that it is in an asymptotic direction. Thus we want the equations obtained by differentiating (7) once and twice with respect to μ to hold. With a little manipulation these come to (writing X for $x - c_1$ and Y for $y - c_2$)

$$(X, Y, h(x,y) - c_3).(-h_x, -h_y, 1) = 0, \quad (8)$$
$$h_{xx}X^2 + 2h_{xy}XY + h_{yy}Y^2 = 0. \quad (9)$$

Here, h and its derivatives are evaluated at $(x(t), y(t))$. Of course, (8) says merely that $\mathbf{r} - \mathbf{c}$ is perpendicular to the normal to M, which is the contour generator condition.

4.2 A special example

Let us apply the above to the surface M with equation

$$z = h(x,y) = -xy + \frac{1}{3}(x^3 + y^3).$$

Let us further consider the straight line motion $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t)) = (1 + t, 2t, 3t)$. See Figure 3. We show how to use the formula (4) above to obtain the Gauss curvature of M at the origin. Note that this curvature is actually -1 from the equation of the surface. In the next subsection we present a simulated experiment based on the same surface.

First we use (8), (9) to find out about the cusp generator curve on M close to the origin. Consider a curve in the x, y parameter plane, passing through the origin:

$$x = x_1t + x_2t^2 + \ldots, \quad y = y_1t + y_2t^2 + \ldots.$$

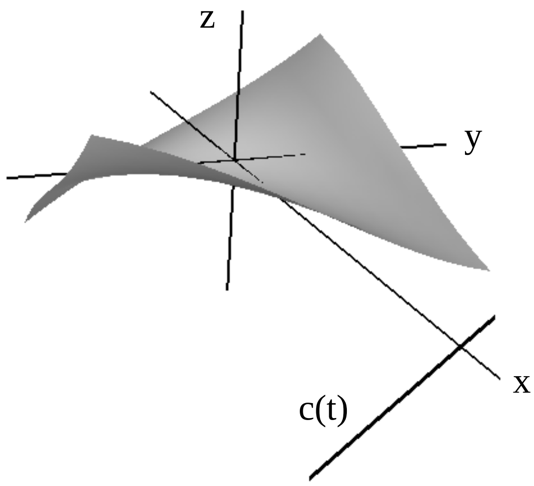

Figure 3: Graph of $h(x,y) = -xy + (1/3)(x^3 + y^3)$ and $\mathbf{c}(t) = (1 + t, 2t, 3t)$

We substitute these into (8). Comparing coefficients of t gives $y_1 = -3$ and comparing coefficients of t^2 gives $y_2 = x_1^2 - 8x_1 + 3$. That is, y_1 is fixed but we can choose x_1 and then deduce y_2. If we use (9) as well, then the curve on M is determined uniquely; the expansion of (x, y) starts off

$$(5t + 32t^2 + \ldots, \; -3t - 12t^2 + \ldots).$$

But to use the formula (4) for K (and the corresponding formula for the mean curvature) *we do not need no know any more than* y_1*, which comes from the condition that* $\mathbf{r}(t)$ *lies on the contour generator* $\Sigma(t)$*.*

In fact, the image $\mathbf{p}(t)$ of the curve $\mathbf{r}(t)$ comes to

$$(-1 + \ldots, \quad -5t + \ldots, \quad -3t + \ldots),$$

where in each case $\ldots$ stands for terms of degree 2 or higher. This uses only $y_1 = -3$, *not* the extra conditions obtained from (9). Therefore, at $t = 0$, where we have the surface normal $\mathbf{n}$ equal to $(0, 0, 1)$, we have $\mathbf{p}_t.\mathbf{n} = -3$. Also $\mathbf{c}_t = (1, 2, 3)$ (for any t in our example), and $\mathbf{p}(0) = (-1, 0, 0)$, so $[\mathbf{p}, \mathbf{c}_t, \mathbf{p}_t] = -9$ and (4) gives $K = -(-3)^4/(-9)^2 = -1$.

In the general argument of the Appendix, we simplify the calculation a little by using, instead of the image sphere, an image plane tangent to this sphere at $(-1, 0, 0)$, but the principle of the calculation is identical to that in the example just given.

4.3 A simulated experiment

We now present an example in the form of a simulated experiment. This has the advantage that we *know* the Gauss curvature and depth of the test surface M, so can check on the accuracy of the results

Time t	Measured K	Actual K	Measured λ	Actual λ
0.00	-1.08	-1.00	0.99	1.00
-.02	-1.00	-1.00	1.10	1.08
-.04	-1.03	-0.99	1.14	1.16
-0.06	-1.09	-0.98	1.25	1.25

Table 1: Values of K and λ calculated from the data and compared with exact values (2 d.p's).

obtained by using the depth formula (2) and the formula (4) for K above. In these formulae we will estimate the derivative $\mathbf{p}_t$ by taking values at discrete time intervals.

We use a simple difference algorithm to estimate the derivative. In fact we take five consecutive positions on the image curve $\mathbf{p}(t)$, given by $t = t_0 - 2h, t_0 - h, t_0, t_0 + h, t_0 + 2h$, approximate the image curve C by a polynomial curve C' of degree 4, and use the derivative of C' at $t = 0$ as our approximation. This gives the approximate formula for $\mathbf{p}_t(t_0)$:

$$\frac{\mathbf{p}(t_0 - 2h) - 8\mathbf{p}(t_0 - h) + 8\mathbf{p}(t_0 + h) - \mathbf{p}(t_0 + 2h)}{12h}$$

where h is the time difference between observed cusp points.

We use the same example as in §3, namely M is the graph of the function $h(x, y) = -xy + (1/3)(x^3 + y^3)$ and the camera motion we take is the straight line trajectory, $\mathbf{c}(t) = (1 + t, 2t, 3t)$ (Figure 3). Thus $\mathbf{c}_t$ is the constant vector (1,2,3).

Table 1 shows the results of our calculations for some different times, and gives the exact analytically calculated values.

5 General motion constraint

Here, we show that in a certain precise sense there is no general constraint on the motion obtainable from following cusps (see the remark at the end of this section). In fact we show that, using the locus of cusps as a parametrized curve $\mathbf{q}(t)$ (using rotated coordinates) in the sphere, and using also the normal lines to the cusps, there cannot be any constraint on the motion. Explicitly, we claim the following, where t is a real number lying in some (small) open interval $t_1 < t < t_2$.

Theorem: *Suppose that $\mathbf{q}(t), \mathbf{n}(t)$ are given smooth families of orthogonal unit vectors, that $R(t)$ is a smooth family of 3-dimensional rotations, and that $\mathbf{c}(t)$ is a smooth space curve. Then we can find a smooth surface M in 3-space for which $\mathbf{q}(t)$ is the locus of cusps of apparent contours arising from camera centres $\mathbf{c}(t)$, with rotated coordinates $\mathbf{q}$ ($\mathbf{p} = R(\mathbf{q})$ in the usual notation) and $R(t)(\mathbf{n}(t))$ is the normal to the apparent contour at the cusp point.*

Proof Let $\mathbf{p}(t) = R(t)\mathbf{q}(t)$ and replace also $\mathbf{n}$ by its unrotated form $R(t)\mathbf{n}(t)$ (we shall continue to use $\mathbf{n}$). We then seek a surface M with the following properties:

- for each t, there is a point $\mathbf{r}(t) = \mathbf{c}(t) + \lambda(t)\mathbf{p}(t)$ on M for some $\lambda(t)$,
- the normal to M at $\mathbf{r}(t)$ is $\mathbf{n}(t)$,
- for each t, the vector $\mathbf{p}(t)$ is in an asymptotic direction at $\mathbf{r}(t)$ (this ensures that the apparent contour at 'time' t has a cusp at the apparent contour point $\mathbf{c}(t) + \mathbf{p}(t)$).

There is no choice for the function λ, since we require (using subscripts to denote differentiation as usual) $\mathbf{r}_t = \mathbf{c}_t + \lambda\mathbf{p}_t + \lambda_t\mathbf{p}$, and since $\mathbf{n}(t)$ is required to be normal to the surface, we deduce the usual formula $\lambda(t) = -\mathbf{c}_t.\mathbf{n}/\mathbf{p}_t.\mathbf{n}$, noting here that $\mathbf{p}$ is a function of *one* variable t, since it gives the position of the cusp (in unrotated $\mathbf{p}$ coordinates).

We now have a space curve $\mathbf{r}(t)$, and, along that curve, we shall require our surface M to have normal $\mathbf{n}(t)$ (for this is parallel to the apparent contour normal in the unrotated coordinates). This gives us a 'surface strip' in the language of Koenderink [12].

The final requirement on M is that, at each point $\mathbf{r}(t)$, an asymptotic direction is in the specified direction $\mathbf{p}(t)$. This amounts to saying that, in the direction $\mathbf{p}(t)$, the sectional curvature of M is zero, that is the section of M by the plane through $\mathbf{r}(t)$ containing $\mathbf{p}(t)$ and $\mathbf{n}(t)$ has an (ordinary) inflexion at $\mathbf{r}(t)$. There is no difficulty in constructing an M with this property, so long as the asymptotic direction does not actually coincide with the tangent to the curve $\mathbf{r}(t)$. But in that case it is easy to check that the locus of cusps $\mathbf{p}(t)$ in the image sphere would be singular.

Remark The last two formulae in Proposition 3.1 describe the relative motion between (i) the image of a fixed surface marking and (ii) the locus of cusps, or indeed the image of any other locus on the surface provided this is parametrised by t and starts at a cusp point in the image. We can use these formulae, and the others in Proposition 3.1 to derive a constraint on motion. See [5].

A Appendix:Sketch of Proof of Equation (1) and Proposition 3.1

First, we prove (1). At nonsingular points of the apparent contour the tangent direction is that of $\mathbf{p}_u$, or of the unit vector $\widehat{\mathbf{p}_u} = \mathbf{p}_u / \parallel \mathbf{p}_u \parallel$. Note that at an ordinary cusp the limit of this *does* exist. By l'Hôpital's rule, the limit coincides with the limit of $\mathbf{p}_{uu}/\widehat{\mathbf{p}_u}.\mathbf{p}_{uu}$, the denominator here being the derivative of $\parallel \mathbf{p}_u \parallel$. Now at an ordinary cusp, by definition, the second and third derivatives $\mathbf{p}_{uu}, \mathbf{p}_{uuu}$ are independent (compare [2, p.155]), and in particular $\mathbf{p}_{uu}$ is non-zero. Thus the limit of the above expression has direction that of $\mathbf{p}_{uu}$, i.e., the limiting tangent has the latter direction. Hence the limiting normal is in the direction $\mathbf{p} \wedge \mathbf{p}_{uu}$.

Results like Proposition 3.1 can be readily proved by setting up the surface in 'Monge form', that is as a graph $z = h(x, y)$ with the tangent plane at the origin coinciding with the x, y plane, i.e. $h = h_x = h_y = 0$ at $x = y = 0$. (Compare Figure 3.) We can in the present case also assume that the x-axis is along an asymptotic direction at the origin, and that our camera centre is initially situated on the x-axis, at $(1, 0, 0)$ say, so that the apparent contour does have a cusp at time zero. This amounts to saying that the surface can be taken to have the form

$$z = h(x, y) = b_1xy + b_2y^2 + \text{ higher order terms.}$$

In particular the Gauss curvature K at the origin is $-b_1^2$. The camera centres $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$ can be taken as

$$(\lambda_0 + c_{11}t + \ldots, c_{21}t + \ldots, c_{31}t + \ldots).$$

Because coordinates in the image sphere are difficult to work with, it is advantageous to project the sphere of radius 1 centred at the origin, outwards from the centre, on to the plane $x = -1$, by the map $(x, y, z) \to (-y/x, -z/x)$. For $t = 0$ our vector $\mathbf{p}$ will be in the direction $\mathbf{r} - \mathbf{c}$, which is $-\mathbf{c} = (-\lambda_0, 0, 0)$. Hence the initial value of the unit vector $\mathbf{p}$ is $(-1, 0, 0)$. Taking any curve $\mathbf{p}(t) = (p_1(t), p_2(t), p_3(t))$ on the sphere, passing through $(-1, 0, 0)$ when $t = 0$, we can project this outwards on to the plane $x = -1$. Furthermore, it is easy to check that the first two derivatives of $\mathbf{p}$ at $t = 0$ are precisely the same as the first two derivatives at 0 of the planar curve. Note that if we start with a curve $\mathbf{r}(t) = (x(t), y(t), h(x(t), y(t))$ on M, find the corresponding image curve $\mathbf{p}(t) = (\mathbf{r}(t) - \mathbf{c}(t))/||\mathbf{r}(t) - \mathbf{c}(t)||$ and then project outwards to the plane, the resulting plane curve has the reasonably simple form

$$\left(\frac{c_2(t) - y(t)}{x(t) - c_1(t)}, \frac{c_3(t) - h((x(t), y(t))}{x(t) - c_1(t)}\right).$$

When we take a curve $x(t) = x_1t + \ldots, y(t) = y_1t + \ldots, z = h(x, y)$ on the surface M, we can write down the condition that the point $\mathbf{r}(t)$ of this curve with parameter t lies on the contour generator of the motion at time t, i.e. that $(\mathbf{r}(t) - \mathbf{c}(t)).\mathbf{n}(t) = 0$ where $\mathbf{n} = (-h_x, -h_y, 1)$ is a normal vector to M. Comparing coefficients of powers of t this gives rise to a sequence of equations which express the coefficients $x_1, y_1, \ldots$ of the curve in terms of surface and motion coefficients. These equations begin with $y_1 = c_{31}/(\lambda_0 b_1)$, and in fact this is the only one needed here. The normal $\mathbf{n}$ and tangent $\mathbf{t}$ at $t = 0$ can be taken as $(0, 0, 1)$ and $(0, -1, 0)$ respectively. The derivative of the image curve at $t = 0$ comes to

$$\mathbf{p}_t(0) = \left(\frac{c_{31}}{\lambda_0^2 b_1} - \frac{c_{21}}{\lambda_0}, \frac{-c_{31}}{\lambda_0}\right). \qquad (10)$$

Furthermore $[\mathbf{p}, \mathbf{c}_t, \mathbf{p}_t]$ at $t = 0$ comes to $\mathbf{c}_{31}^2/(\lambda_0^2 b_1)$, and using $K = -b_1^2$ gives (3). Note that $[\mathbf{p}, \mathbf{c}_t, \mathbf{p}_t]$ has the sign of b_1. The usual depth formula $\lambda = -\mathbf{c}_t.\mathbf{n}/\mathbf{p}_t.\mathbf{n}$ gives (4). Interpreting the terms of (10) gives the other two formulae (5) and (6).

Acknowledgements The computer pictures in Figures 2, and 3 were produced using the 'Liverpool Surfaces Modelling Package' written by Richard Morris. We also acknowledge support through grant GR/H59855 from the British research council EPSRC (formerly SERC). G. Fletcher is supported by EPSRC.

References

[1] K.Åström, R.Cipolla and P.J.Giblin, 'Motion from the frontier of curved surfaces', these *Proceedings.*

[2] J.W.Bruce and P.J.Giblin, *Curves and Singularities*, Cambridge University Press, 2nd Edition 1992.

[3] R.Cipolla and A.Blake, 'The dynamic analysis of apparent contours' *Proc. 3rd Internat. Conf. on Computer Vision*, pp.616-623, 1990.

[4] R.Cipolla and A.Blake, 'Surface shape from deformation of apparent contours', *Int. J. of Computer Vision* 9 (1992), 83-112.

[5] R.Cipolla and P.Giblin, 'Following cusps', Preprint, 1994.

[6] P.J.Giblin, J.E.Rycroft and F.E.Pollick, 'Recovery of an unknown axis of rotation from the profiles of a rotating surface', J.Opt. Soc. America 11A (1994), 1976-1984.

[7] P.J.Giblin and M.G.Soares, 'On the geometry of a surface and its singular profiles', *Image and Vision Computing* 6(1988), 225–234.

[8] P.J.Giblin and R.S.Weiss, 'Reconstruction of surfaces from profiles', First Internat. Conf. on Computer Vision, London, 1987, pp. 136–144.

[9] P.J.Giblin and R.S.Weiss, 'Epipolar curves on surfaces', to appear in *Image and Vision Computing*; 'Epipolar fields on surfaces', *Proceedings of ECCV, Stockholm 1994, Springer Lecture Notes on Computer Science* 800 (1994), 14–23.

[10] J.J.Koenderink and A.J.Van Doorn, 'The shape of smooth objects and the way contours end', *Perception* 11 (1982), 129–137

[11] J.J.Koenderink, 'What does the occluding contour tell us about solid shape?', *Perception* 13 (1984), 321–330.

[12] J.J.Koenderink, *Solid Shape*, M.I.T.Press 1990.

[13] J.Ponce and A.Noble, 'Reconstructing parametrized surfaces from x-ray projections' Preprint, General Electric Corporate Research and Development Center.

[14] R.Vaillant and O.D.Faugeras, 'Using extremal boundaries for 3D object modelling', *Patt. Recog. and Machine Intell.* 14 (1992), 157–173.