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Following Cusps

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Abstract. It is known that the deformation of the apparent contours of a surface under perspective projection and viewer motion enable the recovery of the geometry of the surface, for example by utilising the *epipolar parametrization*. These methods break down with apparent contours that are singular i.e., with *cusps*. In this paper we study this situation and show how, nevertheless, the surface geometry (including the Gauss curvature and mean curvature of the surface) can be recovered by *following the cusps*. Indeed the formulae are much simpler in this case and require lower spatio-temporal derivatives than in the general case of nonsingular apparent contours. We also show that following cusps does not by itself provide us with information on viewer motion.

1. Introduction

For smooth curved surfaces an important image feature is the profile or apparent contour. This is the projection of the locus of points on the surface which separates the visible and occluded parts. See Fig. 1. Under perspective projection this locus—the critical set or contour generator—can be constructed as the set of points on the surface which are touched by rays through the projection centre. The contour generator is dependent on the local surface geometry (via tangency and conjugacy constraints) and the viewpoint. Each viewpont will generate a different contour generator with the contour generators 'slipping' over the visible surface under viewer motion.

The family of contour generators generated under continuous viewer motion can be used to represent the visible surface. Giblin and Weiss (1987) and Cipolla and Blake (1992) have shown how the spatio-temporal analysis of deforming image apparent contours (*profiles*) enables computation of local surface curvature along the corresponding contour generator (*critical sets*) on the surface, assuming viewer motion is known. To perform the analysis, however, a spatio-temporal parametrization of image-curve motion is needed, but is underconstrained. The *epipolar* parametrization is most naturally matched to the recovery of surface curvature. In this parametrization (for both the spatio-temporal image and the surface), *correspondence* between points on successive snapshots of an apparent contour is set up by matching along epipolar lines. Namely the corresponding ray in the next viewpoint (in an infinitesimal sense), is chosen so that it lies in the epipolar plane defined by the viewer's translational motion and the ray in the first viewpoint. The parametrization leads to simplified expressions for the recovery of depth and surface curvature from image velocities and accelerations and known viewer motion. It is especially suited to the recovery of surface geometry by an active explorer making deliberate viewer motions around an object of interest and it has been successfully implemented in various systems

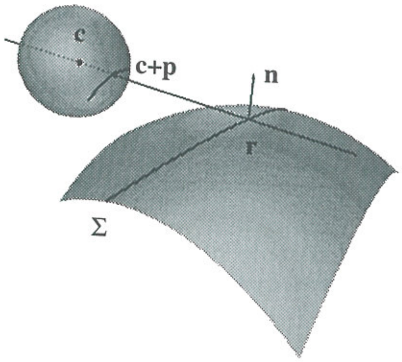

Figure 1. Perspective projection: the contour generator Σ with a typical point **r**, the image sphere with centre **c** and the corresponding apparent contour point **c** + **p**. Thus **p** is the unit vector joining the centre **c** to the apparent contour point.

(Cipolla and Blake, 1992; Vaillant and Faugeras, 1992).

There are however several cases in which this parametrization is degenerate and so can not be used to recover the local surface geometry. The first case of degeneracy occurs when the contour generator does not *slip* over the surface with viewer motion but is fixed to it. This is the case of viewing a 3D rigid curve attached to the surface such as a surface marking. In this case the epipolar parametrization successfully allows the recovery of the structure of a space curve from image velocities (it is analogous to stereo reconstruction in the infinitesimal limit) but the surface orientation is no longer completely defined but constrained to be perpendicular to the curve tangent. This case poses no special problems. In fact one advantage of the epipolar parametrization is that it leads to a uniform treatment to the recovery of depth for rigid space curves as well as the occluding contours of smooth surfaces. The former can be simply treated as occluding contours with infinite curvature in the direction of the viewing ray—a property which has been successfully used to discriminate between fixed space curves and the occluding contours of smooth surfaces.

Another case of degeneracy occurs at a point of the surface-to-image mapping when a ray is tangent not only to the surface but also to the contour generator. This will occur when viewing a hyperbolic surface patch along an asymptotic direction. For a transparent surface this special point on the contour generator will appear as a *cusp* on the apparent contour. For opaque surfaces, however, only one branch of the cusp is visible and the contour ends abruptly (Koenderink and Van Doorn, 1982; Koenderink, 1984). We call such a surface point a *cusp generator point* and the corresponding image point simply a *cusp point*. See Fig. 2.

The epipolar parametrization of the surface will be degenerate at these cusp generator points since the ray and contour generator are parallel and do not form a basis for the tangent plane. The epipolar spatio-temporal parametrization under viewer motion of the apparent contours can no longer be used to recover depth, surface orientation and surface curvature at these points. Under viewer motion the locus of the cusp generator points on the surface defines the *cusp generator curve*. In the vicinity of this surface strip the epipolar parametrization will be ill-conditioned and impractical.

The remaining cases of degeneracy of the epipolar parametrization will not concern us here—they come from singular contour generators ('lips/beaks' transitions) and frontier points (where the epipolar plane is tangent to the surface). See (Giblin and Weiss, 1995; Giblin et al., 1994) for further details.

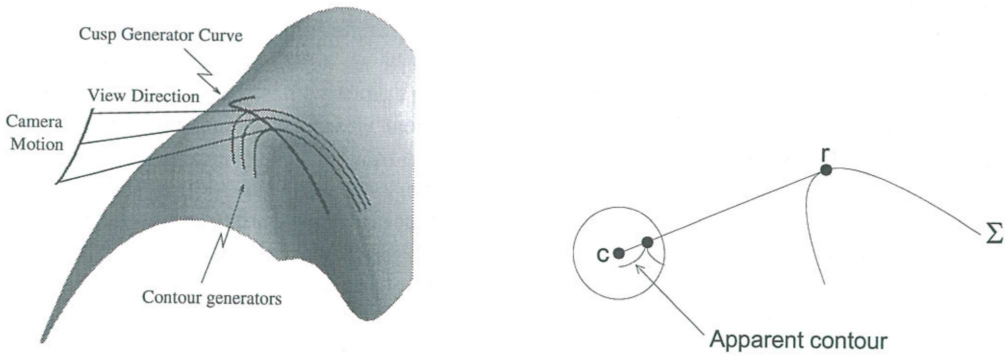

Figure 2. Left: three viewing positions $\mathbf{c}(t)$, all of which produce cusps on the apparent contours because some viewline (ray) from $\mathbf{c}(t)$ is tangent to the corresponding contour generator Σ. The cusp generator curve is labelled L. Right: A single contour generator Σ as in the figure above produces a cusp on the apparent contour when the viewline is tangent to Σ.

Although the cusp points are difficult to detect by photometric methods in video images of opaque objects they are visible in the images of transparent objects and in X-ray imaging. This is not to say that they are easy to locate accurately: cusps tend to occur as a dark blur even in an X-ray image. (There is such an image in (Koenderink, 1990, p. 425).) But at least in principle, cusp points offer the possibility of being detected and tracked under viewer motion. Giblin and Soares (1988) presented a first attempt to relate local surface geometry (Gaussian and mean curvatures and principal directions) to the image motion of cusps under orthographic projection and planar viewer motion. We extend this here to arbitrary nonplanar, curvilinear viewer motion under perspective projection. We show how the image motion of the cusp can be used to induce an alternative parametrization of the surface in the vicinity of the cusp generator which can be used to recover surface depths and orientation. Remarkably this leads to simplified formulae for surface curvature which require only first-order temporal derivatives. Furthermore our simulations suggest that the formulae are fairly robust. The computation of surface curvature at non-singular apparent contour points requires second order spatial and temporal derivatives.

We also investigate the problems and ambiguities in attempting to recover egomotion from the image motion of cusp points and present the results of some simulated experiments. We indicate how global information can be obtained with certain special classes of surface and give one example; a more detailed treatment of this topic will appear elsewhere (Fletcher and Giblin, 1996). Some of the results in the present paper were announced in (Cipolla et al., 1995).

2. Viewing Geometry and Parametrization of the Surface

2.1. *Spherical Perspective Projection*

Consider the perspective projection of a point on a smooth surface M with position vector $\mathbf{r}$. The direction of a ray to the point on a smooth surface can be represented as a unit vector $\mathbf{p}$ defined by

$$\mathbf{r} = \mathbf{c} + \lambda \mathbf{p} \tag{1}$$

where λ is the distance along the ray to the viewed point and $\mathbf{c}$ is the position of the projection centre of the viewer. This is equivalent to considering the imaging device as a spherical pinhole camera of unit radius. See Fig. 1.

For each viewpoint $\mathbf{c}$, the apparent contour determines a family of rays $\mathbf{p}$ emanating from the projection centre which are tangent to the surface so that

$$\mathbf{p}.\mathbf{n} = 0 \tag{2}$$

where $\mathbf{n}$ is the surface normal.

2.2. *Parametrization Using Contour Generators*

Movement of the viewpoint (projection centre) will produce different contour generators on the surface M. A moving monocular observer with position at time t given by $\mathbf{c}(t)$, will generate a one parameter family of contour generators, indexed by time. It is natural to attempt a parametrization of M which is 'compatible' with the motion of the camera centre, in the sense that contour generators M are parameter curves. That is, 't is to be one of the parameters on M', and so we want there to exist a regular (local) parametrization of M of the form $(u, t) \to \mathbf{r}(u, t)$, the set of points $\mathbf{r}(u, t_0)$, for fixed t_0, being the contour generator from viewpoint $\mathbf{c}(t_0)$. The set of points $\mathbf{p}(u, t_0)$ is the corresponding apparent contour in the unit sphere at the origin; the actual apparent contour points in space are $\mathbf{c}(t_0) + \mathbf{p}(u, t_0)$.

Note that (1) and (2) become

$$\begin{aligned} \mathbf{r}(u, t) &= \mathbf{c}(t) + \lambda(u, t)\mathbf{p}(u, t), \\ \mathbf{p}(u, t).\mathbf{n}(u, t) &= 0. \end{aligned} \tag{3}$$

The conditions for such a parametrization to be possible are that

- $\mathbf{r}$ is not a *frontier point*, i.e., an *epipolar tangency point*. At a frontier point the epipolar plane (spanned by the velocity vector $\mathbf{c}_t(t)$ of the camera centre and the viewline $\mathbf{r} - \mathbf{c}$) coincides with the tangent plane to M. This causes the contour generators to form an envelope on M; see (Giblin and Weiss, 1995). We shall assume $\mathbf{c}_t(t).\mathbf{n} \neq 0$, which implies $\mathbf{c}_t(t)$ is not in the tangent plane to M: this rules out frontier points.
- The contour generators are nonsingular curves.

In this paper we are chiefly concerned with singular apparent contours, so we are avoiding only the situations of a 'cusp on the frontier' and of 'lips/beaks' singularities (Koenderink, 1990, p. 458). In the latter case, we can expect cusps to appear or disappear.