Part IA Engineering

Mathematics

Lent Term

Convolution

Fourier Series

Probability

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Contents & Examples Questions

Contents & Examples Qs contd.

Section 1

Linear System & Impulse Response

We motivate the study of linear time-invariant systems.

The principle of superposition is explained.

Step functions and delta functions are introduced, together with their corresponding responses.

Examples are given to illustrate the use of the step response with superposition.

The sifting theorem is stated and illustrated with some examples.

Motivation

Many engineering problems concern linear systems.

Forces → **System** → **Strains**

Voltages → **System** → **Currents**

Pressure → **System** → **Density**

Heat flow → **System** → **Temperature**

Power → **System** → **Kinetic Energy**

In a linear system the output is computed as some linear combination of the inputs (including inputs from the past, if we are considering a system with a time-varying input and output).

Linear Systems

f(t) → **Linear System** → **y(t)**

1. Linear time-invariant systems satisfy the principle of superposition.

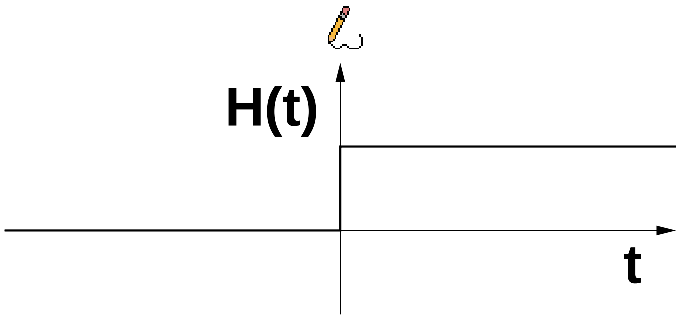
If input $f_1(t)$	$\rightarrow$	output $y_1(t)$
and input $f_2(t)$	$\rightarrow$	output $y_2(t)$
	then	
input $\alpha f_1(t) + \beta f_2(t)$	$\rightarrow$	output $\alpha y_1(t) + \beta y_2(t)$

where α and β are any constants.

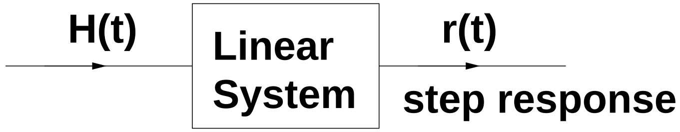
2. Linear systems have the special property that a sine wave at the input leads to a (possibly different) sine wave at the output.

Sine wave → **Linear System** → **Sine wave with amplitude or phase changed**

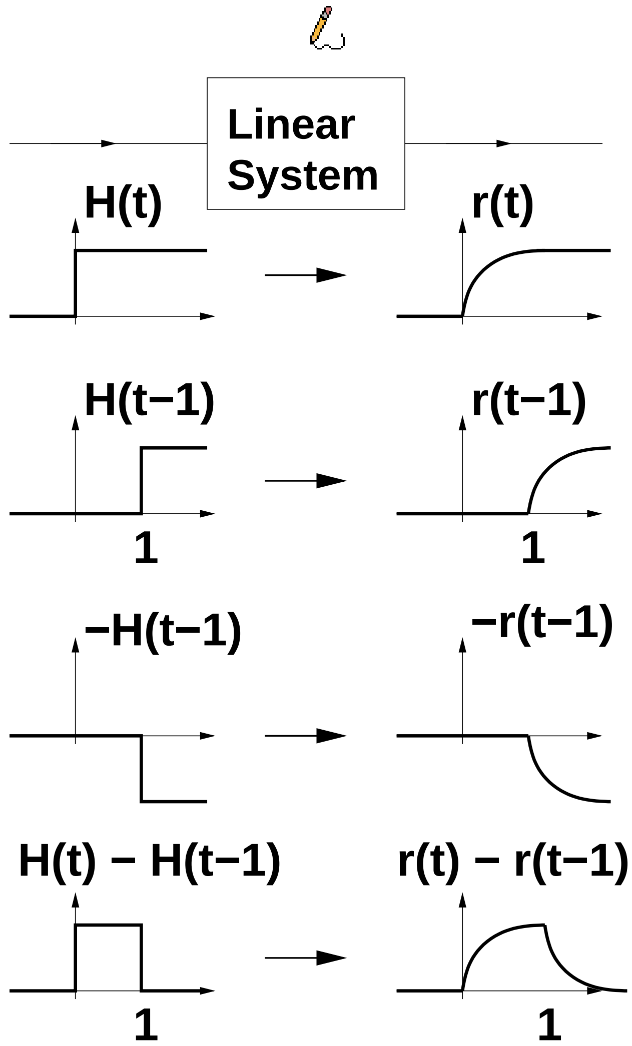
Step Function



$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



Superposition Example

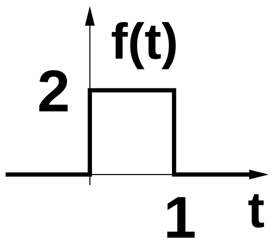


Calculation of Superposition

Find the output of a linear system with step response

$$r(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-5t}, & t \geq 0 \end{cases}$$

when the input is the pulse $f(t)$.



Case (a): When $0 \leq t < 1$ the input is the same as a scaled step function so the output $y(t)$ is given by

$$y(t) = 2\left(1 - e^{-5t}\right) \qquad ,0 \leq t < 1$$

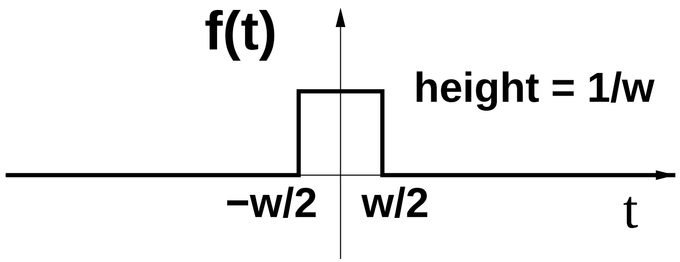
Case (b): When $1 \leq t$

$$f(t) = 2H(t) - 2H(t-1)$$

therefore the output $y(t)$ is given by

$$\begin{aligned} y(t) &= 2r(t) - 2r(t-1) \\ &= 2\left(1 - e^{-5t} - 1 + e^{-5(t-1)}\right) \\ &= 2\left(e^{5} - 1\right)e^{-5t} \qquad ,1 \leq t \end{aligned}$$

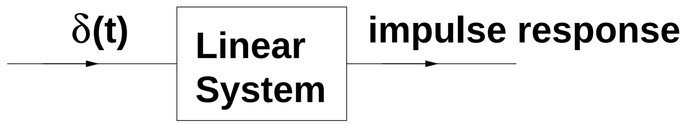
Dirac Delta Function



As $w \rightarrow 0$ the pulse $f(t)$ becomes narrower and taller. In the limit as $w \rightarrow 0$ the pulse $f(t)$ becomes a delta function: $\delta(t)$.

The delta function is a spike with unit area. It goes bang when its argument is zero.

$$\begin{aligned} \delta(t) &= 0 \quad \text{except at } t = 0 \\ \int_a^b \delta(t)dt &= 1 \quad \text{provided } a < 0 \text{ and } b > 0 \end{aligned}$$



Integrating the Delta Function

From the previous page

$$\int_a^b \delta(t)\,dt = 1 \quad \text{provided } a < 0 \text{ and } b > 0$$

thus

$$\begin{aligned} \int_{-\infty}^{T} \delta(t)\,dt &= \begin{cases} 0, & T < 0 \\ 1, & T > 0 \end{cases} \\ &= H(T) \end{aligned}$$

The integral of a delta function is a step function.

Conversely, the derivative of a step function is a delta function.

$$\begin{tikzcd} \textbf{Delta Function } \boldsymbol{\delta(t)} \arrow[r, shift left, "\textbf{integrate}"] & \textbf{Step Function } \mathbf{H(t)} \arrow[l, shift left, "\textbf{differentiate}"] \end{tikzcd}$$

Impulse Response

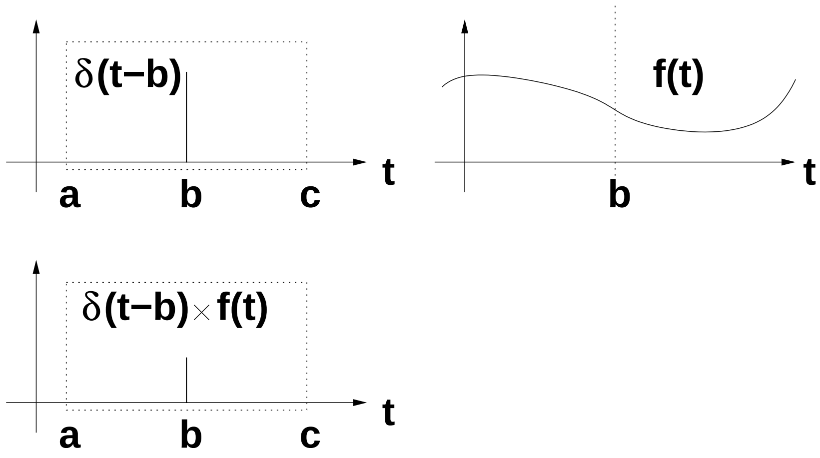
$$\begin{tikzcd} \textbf{Impulse Response } \mathbf{g(t)} \arrow[r, shift left, "\textbf{integrate}"] & \textbf{Step Response } \mathbf{r(t)} \arrow[l, shift left, "\textbf{differentiate}"] \end{tikzcd}$$

Find the output, $g(t)$ of a linear system with step response $r(t) = 1 - e^{-5t}$ when the input is the delta function $\delta(t)$.

$$\begin{aligned} r(t) &= 1 - e^{-5t} \\ g(t) &= \frac{dr}{dt} = 5e^{-5t} \end{aligned}$$

So the impulse response of the system is $5e^{-5t}$.

Sifting Theorem



$$\int_a^c \delta(t-b)dt \;=\; 1 \quad \text{provided } a < b \text{ and } c > b$$

$$\int_a^c \delta(t-b)f(t)dt \;=\; f(b) \quad \text{provided } a < b \text{ and } c > b$$

Sifting Examples

$$\int_{-\pi}^{\pi} \cos(2t)\,\delta(t)\,dt \;=\; \cos(0) = 1$$

$$\int_{-\pi}^{\pi} \cos(2t)\,\delta\left(t-\frac{\pi}{2}\right)\,dt \;=\; \cos(\pi) = -1$$

$$\int_{-\pi}^{0} \cos(2t)\,\delta\left(t-\frac{\pi}{2}\right)\,dt \;=\; 0$$

$$\int_{-\pi}^{\pi} t\,\delta\left(t+\frac{\pi}{2}\right)\,dt \;=\; -\frac{\pi}{2}$$

$$\int_{0}^{\pi} t\,\delta\left(t+\frac{\pi}{2}\right)\,dt \;=\; 0$$

Section 1: Summary

Superposition:

If input $f_1(t)$ $\rightarrow$ output $y_1(t)$
and input $f_2(t)$ $\rightarrow$ output $y_2(t)$ then
input $\alpha f_1(t) + \beta f_2(t)$ $\rightarrow$ output $\alpha y_1(t) + \beta y_2(t)$
where α and β are any constants.

Sifting:

$$\int_a^c \delta(t-b) f(t) dt = f(b) \quad \text{provided } a < b \text{ and } c > b$$

Step function and step response.

Impulse function and impulse response.

Finding the system response to a pulse by combining scaled and delayed step responses using superposition.

Section 2

Differential Equations to

Describe Linear Systems

We motivate the convolution integral, which will be presented in section 3, using an example of a car going up a step.

A technique is described for solving a linear differential equation to obtain the step response of the system. We set the input to 1, and solve with initial conditions $y = \dot{y} = 0$ for $t = 0$. The impulse response can then be obtained by differentiating the step response.

The utility of this technique, when used together with convolution, is outlined.

Differential Equations

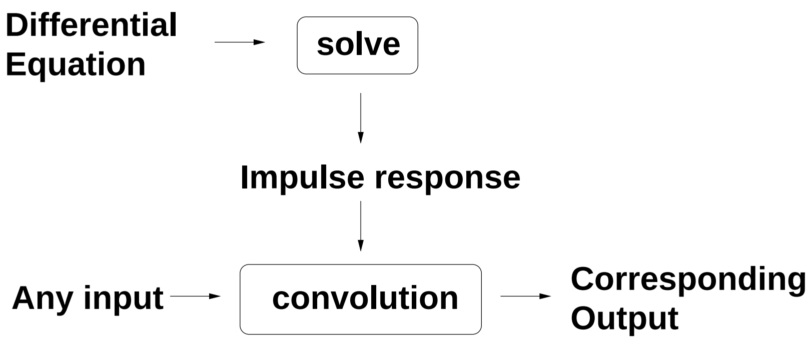
Linear systems are often described using differential equations. For example:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = f(t)$$

where f(t) is the input to the system and y(t) is the output.

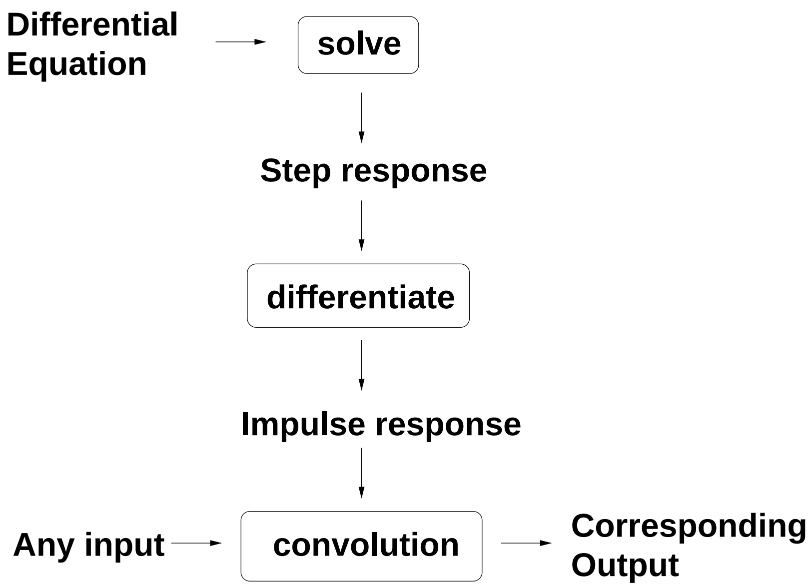
We know how to solve for y given a specific input f.

We now cover an alternative approach:



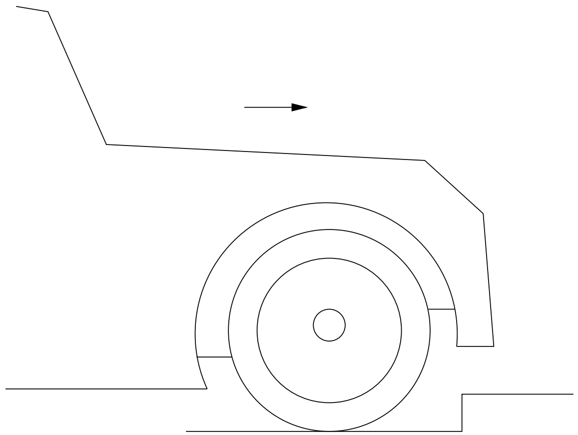
Solving for Impulse Response

We cannot solve for the impulse response directly so we solve for the step response and then differentiate it to get the impulse response.



Motivation: Convolution

If we know the response of a linear system to a step input, we can calculate the impulse response and hence we can find the response to any input by convolution.



Suppose we want to know how a car's suspension responds to lots of different types of road surface.

We measure how the suspension responds to a step input (or calculate the step response from a theoretical model of the system).

We can then find the impulse response and use convolution to find the car's behaviour for any road surface profile.

Solving for Step Response

If we want to find the step response of

$$\frac{dy}{dt} + 5y = f(t)$$

where f is the input and y is the output. It would be nice if we could put $f(t) = H(t)$ and solve. Unfortunately we don't know of a way to do this directly. So we

1. set $f(t) = 1$, and solve for just $t \geq 0$

2. set the boundary condition $y(0) = 0$ (also $\dot{y}(0) = 0$ for second order equations) to imply that $f(t)$ was zero for all $t < 0$.

We thus have a complete solution because $y = 0$ for $t < 0$, and we have found y for all $t \geq 0$.

Boundary Condition Justification

Prove that $y = 0$ at $t = 0$ by contradiction.

We know that $y(t) = 0$ for all $t < 0$. Therefore the only way for y to equal something other than zero at $t = 0$ is if there is a step discontinuity in y at $t = 0$.

Assume that y has a step of height h at $t = 0$. If y has a step discontinuity at $t = 0$ then $\frac{dy}{dt}$ must have a delta function at $t = 0$.

So we have:

- $f(t)$ is a step function so $|f(t)| \leq 1$ for all t.
- $|y| \leq h$ at $t = 0$.
- $\left|\frac{dy}{dt}\right| \to \infty$ at $t = 0$.

Which violates the original equation at $t = 0$.

$$\frac{dy}{dt} = f(t) - 5y$$

As the RHS is finite but the LHS is infinite. Therefore y must be continuous at $t = 0$, and we can use the initial condition $y(0) = 0$.

Step Response Example

Step 1: set $f(t) = 1$, and solve for just $t \geq 0$.

$$\frac{dy}{dt} + 5y = 1$$

Complimentary function: $\dot{y} + 5y = 0 \Rightarrow y = Ae^{-5t}$

Particular Integral: try $y = \lambda$ (a const) $\Rightarrow y = \frac{1}{5}$

General Solution: $y = Ae^{-5t} + \frac{1}{5}$

Step 2: set the boundary condition $y = 0$ at $t = 0$

$y(0) = 0 \Rightarrow A + \frac{1}{5} = 0 \Rightarrow A = -\frac{1}{5}$

So step response is $y(t) = \frac{1}{5}\left(1 - e^{-5t}\right)$ for $t \geq 0$.

Step $\rightarrow$ Impulse Response

Impulse Response g(t) $\underset{\textbf{differentiate}}{\overset{\textbf{integrate}}{\rightleftarrows}}$ **Step Response**

Step response is $y(t) = \frac{1}{5}\left(1 - e^{-5t}\right)$ for $t \geq 0$.

Impulse response $g(t)$ is given by:

$$g(t) = \begin{cases} 0,\ t < 0 \\ \dfrac{d}{dt}\left[\dfrac{1}{5}\left(1 - e^{-5t}\right)\right] = e^{-5t},\ t \geq 0 \end{cases}$$

Find the Impulse Response

$$\frac{d^2y}{dt^2} + 13\frac{dy}{dt} + 12y = f(t)$$

1. Find the General Solution with $f(t) = 1$

Complimentary function is $y = Ae^{-12t} + Be^{-t}$

Particular integral is $y = \frac{1}{12}$

General solution is $y = \frac{1}{12} + Ae^{-12t} + Be^{-t}$

2. Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.

$$\begin{aligned} &\tfrac{1}{12} + A + B = 0 \\ &-12A - B = 0 \\ &\Rightarrow A = \tfrac{1}{132} \text{ and } B = -\tfrac{1}{11} \end{aligned}$$

Thus Step Response is $y = \frac{1}{12} + \frac{e^{-12t}}{132} - \frac{e^{-t}}{11}$

3. Differentiate the step response to get the impulse response.

$$g(t) = \frac{dy}{dt} = \frac{e^{-t} - e^{-12t}}{11} \qquad , (t \geq 0)$$

Using the Impulse Response

If we have a system input composed of impulses,

$$f(t) = 3\delta(t-1) + 4\delta(t-2)$$

we can find the corresponding system output using superposition.

For $t \geq 2$

$$\begin{aligned} y(t) &= 3g(t-1) + 4g(t-2) \\ &= 3\left[\frac{e^{-(t-1)} - e^{-12(t-1)}}{11}\right] \\ &\quad +4\left[\frac{e^{-(t-2)} - e^{-12(t-2)}}{11}\right] \end{aligned}$$

More General Input

Suppose our input is composed of lots of delta functions:

$$f(t) = \sum_n p_n\,\delta(t-q_n)$$

Then the corresponding system output will be

$$y(t) = \sum_n p_n\,g(t-q_n)$$

Section 2: Summary

Differential Equation
$a\ddot{y} + b\dot{y} + cy + d = f(t)$

$\downarrow$

solve
$a\ddot{y} + b\dot{y} + cy + d = 1$
with boundary conditions
$y(0) = 0$ and $\dot{y}(0) = 0$

$\downarrow$

Step response

$\downarrow$

differentiate

$\downarrow$

Impulse response

$\downarrow$

Any input $\rightarrow$ **convolution** $\rightarrow$ **Corresponding Output**

Section 3

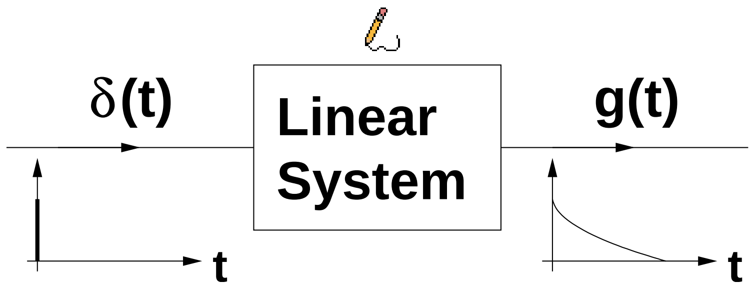
Convolution

In this section we derive the convolution integral and show its use in some examples.

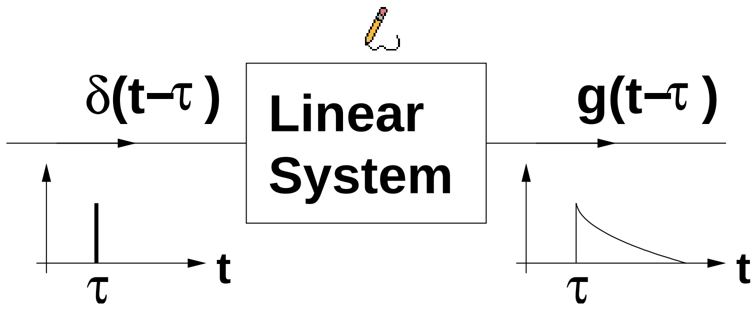
Convolution

Our goal is to calculate the output, $y(t)$ of a linear system using the input, $f(t)$, and the impulse response of the system, $g(t)$.

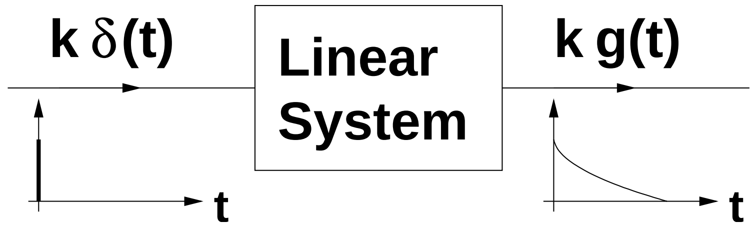
An impulse at time $t = 0$ produces the impulse response.



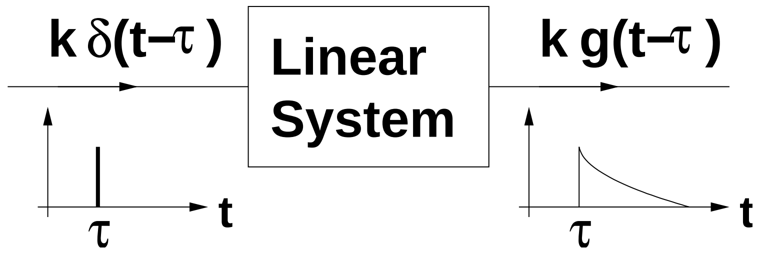
An impulse delayed to time $t = \tau$ produces a delayed impulse response starting at time τ.



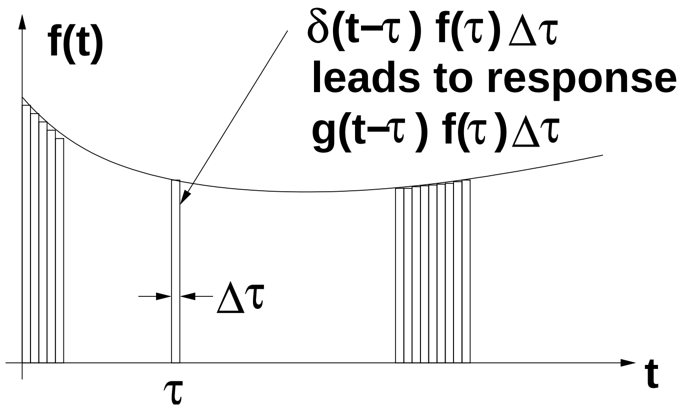
A scaled impulse at time $t = 0$ produces a scaled impulse response.



An impulse that has been scaled by k and delayed to time $t = \tau$ produces an impulse response scaled by k and starting at time τ.



Consider the input, $f(t)$ to be made up of a sequence of strips of width $\Delta\tau$. Each of these strips is similar to a delta function and thus leads to a system output of an appropriately scaled and delayed impulse response.



The response of the system, $y(t)$ is thus the sum of these delayed, scaled impulse responses. (Provided $g(t) = 0$ for $t < 0$.)

$$y(t) \approx \sum_{\substack{\text{All}\\ \text{slices}}} g(t-\tau)f(\tau)\Delta\tau$$

Let the width of the slices tend to zero. The sum turns into an integral called the convolution integral.

$$y(t) = \int_{-\infty}^{t} g(t-\tau)f(\tau)d\tau$$

$$y(t) = \int_{-\infty}^{t} g(t-\tau)f(\tau)d\tau$$

- Treat t as a constant when evaluating the integral. The integration variable is τ.
- t is time as it relates to the output of the system $y(t)$.
- τ is time as it relates to the input of the system $f(\tau)$.

Convolution Example 1

Consider a system with impulse response

$$g(t) = \begin{cases} 0 & , t < 0 \\ e^{-5t} & , t \geq 0 \end{cases}$$

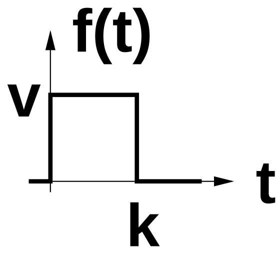
Find the output for input $f(t) = H(t)$ (step function).

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau \\ &= \int_{-\infty}^{t} e^{-5(t-\tau)} H(\tau) d\tau \\ &= \int_{0}^{t} e^{-5(t-\tau)} d\tau \\ &= \left[\frac{1}{5} e^{-5(t-\tau)} \right]_0^t \\ &= \frac{1}{5}\left(1 - e^{-5t}\right) \end{aligned}$$

Convolution Example 2

For the same system ($g(t) = e^{-5t}$, $t \geq 0$), find the output for input

$$f(t) = \begin{cases} 0, & t < 0 \\ v, & 0 < t < k \\ 0, & t > k \end{cases}$$



Using the convolution integral, the answer is given by

$$y(t) = \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau$$

$$= \begin{cases} \int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau, & t < 0 \\ \\ \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\ \quad + \int_0^t g(t-\tau)\, v \, d\tau, & 0 < t < k \\ \\ \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\ \quad + \int_0^k g(t-\tau)\, v \, d\tau \\ \quad + \int_k^t g(t-\tau) \times 0 \, d\tau, & t > k \end{cases}$$

Case (a): $t < 0$

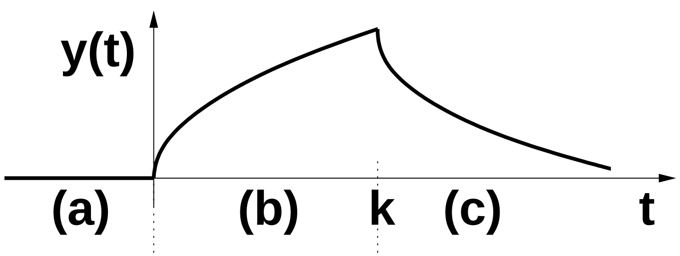
$\int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau = 0$ so $y(t) = 0$ for all $t < 0$.

Case (b): $0 < t < k$

$$\begin{aligned} y(t) = \int_0^t g(t-\tau)\, v \, d\tau \;&=\; \int_0^t e^{-5(t-\tau)}\, v\, d\tau \\ &=\; \frac{v}{5}\left[e^{-5(t-\tau)}\right]_0^t \\ &=\; \frac{v}{5}\left(1 - e^{-5t}\right) \end{aligned}$$

Case (c): $t > k$

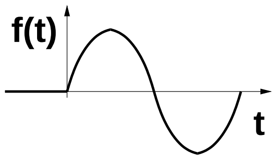
$$\begin{aligned} y(t) = \int_0^k g(t-\tau)\, v \, d\tau \;&=\; \int_0^k e^{-5(t-\tau)}\, v\, d\tau \\ &=\; \frac{v}{5}\left[e^{-5(t-\tau)}\right]_0^k \\ &=\; \frac{v}{5}\left(e^{5k} - 1\right) e^{-5t} \end{aligned}$$



Convolution Example 3

For the same system ($g(t) = e^{-5t}$, $t \geq 0$), find the output for input

$$f(t) = \begin{cases} 0, & t < 0 \\ \sin(\omega t), & t > 0 \end{cases}$$



Using the convolution integral, the answer is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau \\ &= \begin{cases} \int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau, & t < 0 \\ \\ \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\ \quad + \int_0^t g(t-\tau) \sin(\omega\tau) \, d\tau, & 0 < t \end{cases} \end{aligned}$$

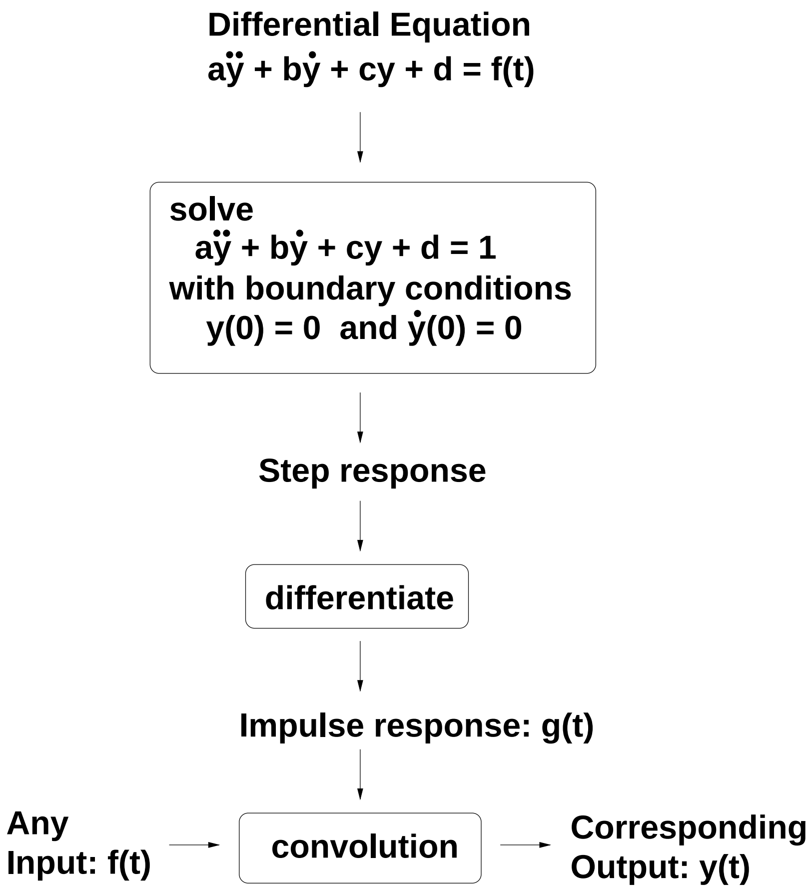
Case (a): $t < 0$

$\int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau = 0$ so $y(t) = 0$ for all $t < 0$.

Case (b): $0 < t$

$$
\begin{aligned}
y(t) &= \int_0^t g(t-\tau)\sin(\omega\tau)\,d\tau \\
&= \int_0^t e^{-5(t-\tau)}\sin(\omega\tau)\,d\tau \\
&= \mathrm{Im}\left\{\int_0^t e^{-5(t-\tau)}e^{i\omega\tau}\,d\tau\right\} \\
&= \mathrm{Im}\left\{e^{-5t}\left[\frac{e^{(5+i\omega)\tau}}{5+i\omega}\right]_0^t\right\} \\
&= \mathrm{Im}\left\{\frac{e^{i\omega t}-e^{-5t}}{5+i\omega}\right\} \\
&= \frac{5\sin(\omega t)-\omega\cos(\omega t)+\omega e^{-5t}}{25+\omega^2}
\end{aligned}
$$

Convolution Summary



$$y(t) = \int_{-\infty}^{t} g(t-\tau)\,f(\tau)\,d\tau$$

Complete Example

Find the impulse response of

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = f(t)$$

hence find the output when the input $f(t) = H(t)e^{-t}$.

1. Find the General Solution with $f(t) = 1$

Complimentary function is $y = Ae^{-t} + Be^{-2t}$

Particular integral is $y = \frac{1}{2}$

General solution is $y = \frac{1}{2} + Ae^{-t} + Be^{-2t}$

2. Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.

$$\begin{array}{l} \frac{1}{2} + A + B = 0 \\ -A - 2B = 0 \\ \Rightarrow A = -1 \text{ and } B = \frac{1}{2} \end{array}$$

Thus Step Response is $y = \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}$

3. Differentiate the step response to get the impulse response.

$$g(t) = \frac{dy}{dt} = e^{-t} - e^{-2t}$$

4. Use the convolution integral to find the output for the required input.

The required input is $f(t) = e^{-t}$, $t > 0$.

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} g(t-\tau)f(\tau)d\tau \\ &= \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) e^{-\tau}\, d\tau \\ &= \int_0^t e^{-t} - e^{\tau-2t}\, d\tau \\ &= \left[\tau e^{-t} - e^{\tau-2t}\right]_0^t \\ &= (t-1)\, e^{-t} + e^{-2t} \end{aligned}$$

Section 3: Summary

Convolution integral (memorise this):

$$
\begin{aligned}
f(t) &= \text{input} \\
g(t) &= \text{impulse response} \\
y(t) &= \text{output} \\
y(t) &= \int_{-\infty}^{t} g(t-\tau)\, f(\tau)\, d\tau
\end{aligned}
$$

Way to find the output of a linear system, described by a differential equation, for an arbitrary input:

- Find general solution to equation for input = 1.
- Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.
- Differentiate to get the impulse response.
- Use convolution integral together with the impulse response to find the output for any desired input.

Section 4

Evaluating Convolution Integrals

A way of rearranging the convolution integral is described and illustrated.

The differences between convolution in time and space are discussed and the concept of causality is introduced.

The concept of a spatially-varying impulse is introduced and the section ends with an example of spatial convolution with a spatially-varying impulse response.

Convolution Summary

Differential Equation

$a\ddot{y} + b\dot{y} + cy + d = f(t)$

$\downarrow$

solve

$a\ddot{y} + b\dot{y} + cy + d = 1$

with boundary conditions

$y(0) = 0$ and $\dot{y}(0) = 0$

$\downarrow$

Step response

$\downarrow$

differentiate

$\downarrow$

Impulse response: $g(t)$

$\downarrow$

Any Input: $f(t)$ $\rightarrow$ **convolution** $\rightarrow$ **Corresponding Output: $y(t)$**

$$y(t) = \int_{-\infty}^{t} g(t-\tau)\, f(\tau)\, d\tau$$

Splitting up Integrals

Suppose we have a function:

$$f(t) = \begin{cases} a & , t < 0 \\ b & , 0 < t < k \\ c & , k < t \end{cases}$$

and we want to evaluate the integral $\int_{-\infty}^{t} f(\tau)\, d\tau$, we can split it up as follows:

$$\begin{array}{ll} \int_{-\infty}^{t} a\, d\tau & , t < 0 \\ \int_{-\infty}^{0} a\, d\tau + \int_{0}^{t} b\, d\tau & , 0 < t < k \\ \int_{-\infty}^{0} a\, d\tau + \int_{0}^{k} b\, d\tau + \int_{k}^{t} c\, d\tau & , k < t \end{array}$$

Example

Find the impulse response of

$$\frac{d^2y}{dt^2} + 9y = f(t)$$

hence find the output for (i) input $f(t) = t,\ t > 0$ and (ii) input $f(t) = H(t) - H(t-1)$.

1. Find the General Solution with $f(t) = 1$

Complimentary function is $y = A\cos(3t) + B\sin(3t)$

Particular integral is $y = \frac{1}{9}$

General solution is $y = \frac{1}{9} + A\cos(3t) + B\sin(3t)$

2. Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.

$$\frac{1}{9} + A = 0$$
$$3B = 0$$
$$\Rightarrow A = -\frac{1}{9} \text{ and } B = 0$$

Thus the Step Response is

$$y = \frac{1}{9}(1 - \cos(3t))$$

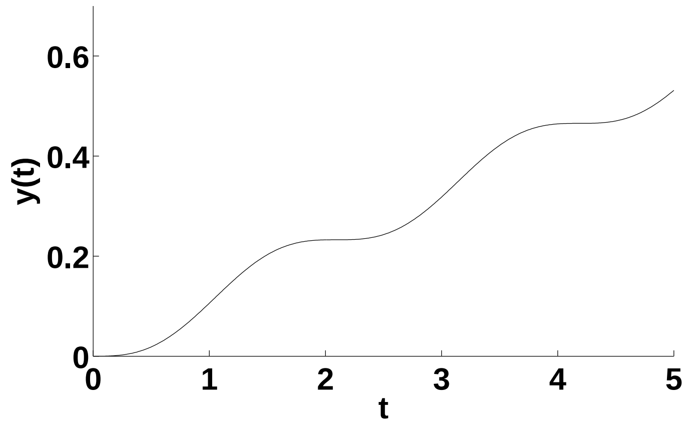
3. Differentiate the step response to get the impulse response.

$$g(t) = \frac{dy}{dt} = \frac{1}{3}\sin(3t)$$

4. Use the convolution integral to find the output for the required input.

For part (i) the required input is a ramp starting at the origin: $f(t) = t$ when $t > 0$ and $f(t) = 0$ otherwise.

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} g(t-\tau) f(\tau)\, d\tau \\ &= \int_0^t \frac{1}{3}\sin(3(t-\tau)) \times \tau\, d\tau \\ &= \frac{t}{9} - \frac{\sin(3t)}{27} \end{aligned}$$



For part (ii) the required input is a pulse of unit height and unit duration: $f(t) = H(t) - H(t-1)$.

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau \\
&= \begin{cases}
\int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau, & t < 0 \\[1em]
\int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\
\quad + \int_{0}^{t} g(t-\tau) \times 1 \, d\tau, & 0 < t < 1 \\[1em]
\int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\
\quad + \int_{0}^{1} g(t-\tau) \times 1 \, d\tau \\
\quad + \int_{1}^{t} g(t-\tau) \times 0 \, d\tau, & t > 1
\end{cases}
\end{aligned}
$$

Case (a): $t < 0$

$\int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau = 0$ so $y(t) = 0$ for all $t < 0$.

Case (b): $0 < t < 1$

$$
\begin{aligned}
y(t) &= \int_{0}^{t} g(t-\tau) \times 1 \, d\tau = \int_{0}^{t} \frac{1}{3} \sin(3(t-\tau)) \, d\tau \\
&= \frac{1}{9}(1 - \cos(3t))
\end{aligned}
$$

Case (c): $1 < t$

$$
\begin{aligned}
y(t) &= \int_{0}^{1} g(t-\tau) \times 1 \, d\tau = \int_{0}^{1} \frac{1}{3} \sin(3(t-\tau)) \, d\tau \\
&= \left[\frac{1}{9} \cos(3(t-\tau)) \right]_{0}^{1} \\
&= \frac{1}{9} \{ \cos(3(t-1)) - \cos(3t) \}
\end{aligned}
$$

Part (ii) Another Way

The input for part (ii) is composed of two step functions. We can therefore calculate the output using the step response, $r(t) = \frac{1}{9}(1 - \cos(3t))$.

Input $= H(t) - H(t-1) \Rightarrow$ Output $= r(t) - r(t-1)$

Hence, for $t > 1$,

$$
\begin{aligned}
y(t) &= \frac{1}{9}(1 - \cos(3t)) - \frac{1}{9}(1 - \cos(3(t-1))) \\
&= \frac{1}{9}\{\cos(3(t-1)) - \cos(3t)\}
\end{aligned}
$$

Alternate Convolution Integral

The normal convolution integral

$$y(t) = \int_{-\infty}^{t} g(t-\tau)\, f(\tau)\, d\tau$$

can be inconvenient to compute when we have a complicated expression for $g(t)$.

We would therefore like to derive an alternative version of the convolution integral that has a term of the form $g(\tau)$ rather than $g(t-\tau)$ as this will be easier to calculate in cases where g is a complicated expression.

Arguments of f and g

Substitute $u = t - \tau$ in the convolution formula. Assume all signals are zero for $t < 0$ and set the lower integral limit to zero. We have $-du = d\tau$,

$$\begin{aligned}\int_0^t g(t-\tau)f(\tau)d\tau &= -\int_t^0 g(u)f(t-u)du \\ &= \int_0^t g(u)f(t-u)du\end{aligned}$$

As u is the variable of integration, we can call it anything, as it disappears when the integration has been evaluated. We therefore choose to rename u as τ. Hence:

$$\int_0^t g(t-\tau)f(\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

So it does not matter which way round you get the arguments to the functions in the convolutions integral, provided both functions are zero for $t < 0$.

Example

Consider a linear system with impulse response

$$g(t) = \begin{cases} 3t^2 - 4t + 7 & , t > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Find the output for the input $f(t) = t$, $(t \geq 0)$ and $f(t) = 0$, $(t < 0)$.

Note that everything is zero for $t < 0$, so we can use $y(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$.

$$\begin{aligned}y(t) &= \int_0^t f(t-\tau)g(\tau)\,d\tau \\ &= \int_0^t (t-\tau) \times (3\tau^2 - 4\tau + 7)\,d\tau \\ &= \frac{t^4}{4} - \frac{2t^3}{3} + \frac{7t^2}{2}\end{aligned}$$

Spatial Convolution

Systems with time-varying input & output.

Causal: no output before the input that causes it.

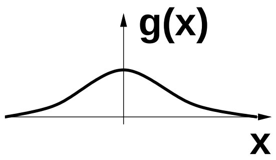
$$g(t) = 0,\ t < 0$$

Systems with input, output a function of position.

An input can affect the output on either side. $g(x)$ can be non-zero for any x.

Consider a one-dimensional strip of a material that is known to deform linearly according to

$$g(x) = \frac{1}{\cosh(x)}$$

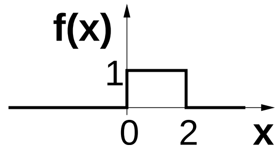


when subject to a unit force at $x = 0$.

This is a spatial impulse response.

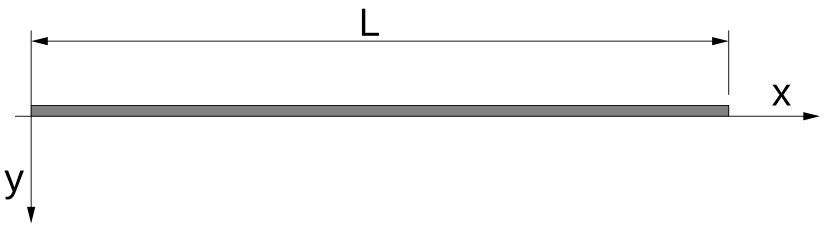
Spatial Convolution Example

Calculate the deformation of a strip of material with spatial impulse response as described on the previous page in response to a uniform load of $f(x) = 1.0$ applied from $x = 0$ to $x = 2$.



$$\begin{aligned}
y(x) &= \int_{-\infty}^{\infty} g(x-\tau) f(\tau)\, d\tau \\
&= \int_{-\infty}^{0} \frac{1}{\cosh(x-\tau)} \times 0\, d\tau \\
&\quad + \int_{0}^{2} \frac{1}{\cosh(x-\tau)} \times 1\, d\tau \\
&\quad + \int_{2}^{\infty} \frac{1}{\cosh(x-\tau)} \times 0\, d\tau \\
&= \int_{0}^{2} \frac{1}{\cosh(x-\tau)}\, d\tau \\
&= 2\left\{\arctan\left(e^{2-x}\right) - \arctan\left(e^{-x}\right)\right\}
\end{aligned}$$

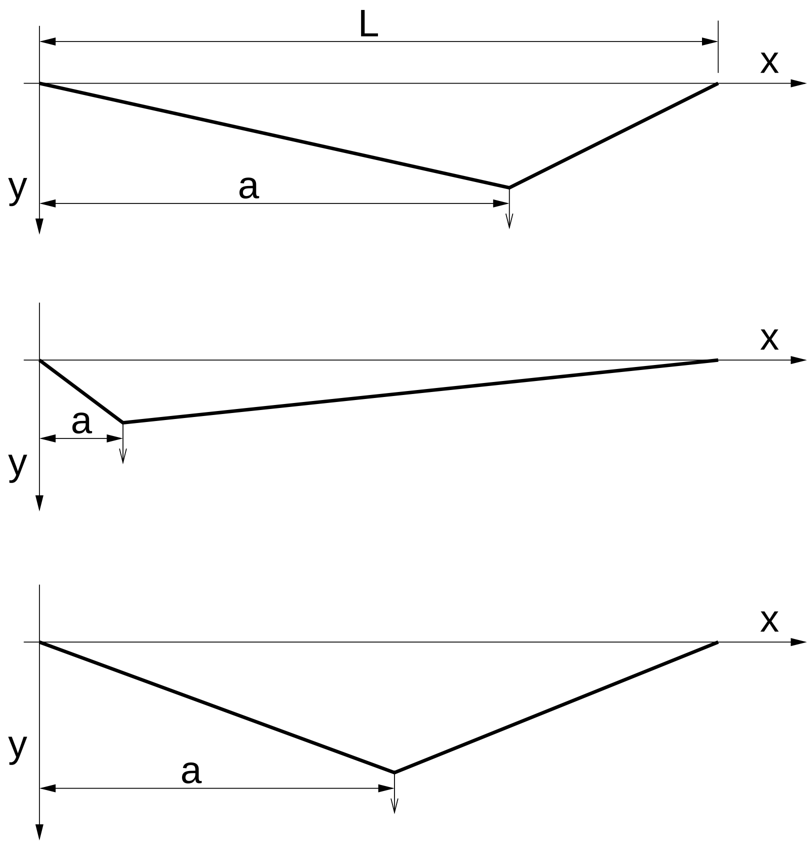
Variable Impulse Response



Consider a taut string suspended between two points a distance L apart. It is subject to a uniform loading of K per unit length which results in a *small* displacement.

If we knew the deformation caused by a point load, we could integrate in a style similar to the convolution integral to find the shape under the distributed load.

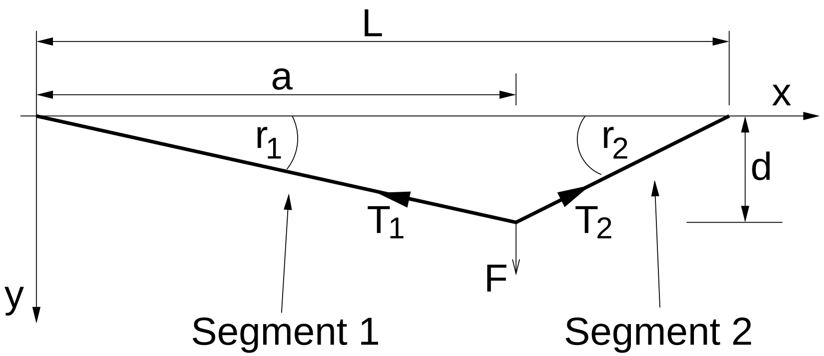
If it was possible to have a spatial impulse response $g(x)$ then we could say $y(x) = \int_0^L g(x-\tau)F(\tau)\,d\tau$



But a normal impulse response is not possible because the shape of g changes depending on the position of the point load along the string. We have a function $g(x,a)$ where the point load is at position a. The function g that gives the displacement under a point load depends on both the position of the load, a, and the position at which you want to know the displacement, x.

If we can find this $g(x,a)$ we can work out the complete displacement under the continuous load K using

$$y(x) = \int_0^L g(x,a)F(a)\,da = \int_0^L g(x,a)K\,da$$



To find $g(x,a)$ we first work out the maximum displacement, d, for a point load, $F = 1$, at position a.

Resolve horisontally: $T_1 \cos(r_1) = T_2 \cos(r_2)$

Use the approximation: $\cos(r_1) \approx \cos(r_2) \approx 1$

This gives us: $T_1 = T_2$. Call this tension T.

Now resolve vertically: $T(\sin(r_1) + \sin(r_2)) = 1$

Again approximate: $\cos(r_1) \approx \cos(r_2) \approx 1$

This gives us: $T(\tan(r_1) + \tan(r_2)) = 1$

$$\begin{aligned}
\Rightarrow \left(\frac{d}{a} + \frac{d}{L-a}\right) &= \frac{1}{T} \\
\Rightarrow \frac{Ld}{a(L-a)} &= \frac{1}{T} \\
\text{so} \qquad d &= \frac{a(L-a)}{TL}
\end{aligned}$$

This enables us to write down equations for the two straight segments of the $g(x,a)$ function.

Segment 1: $x < a$ $\qquad g(x,a) = \left(\frac{x}{a}\right) d = \frac{x(L-a)}{TL}$

Segment 2: $x > a$ $\qquad g(x,a) = \left(\frac{L-x}{L-a}\right) d = \frac{a(L-x)}{TL}$

Finally, we work out the shape of a string of length L with tension T under a uniform load of K per unit length.

$$
\begin{aligned}
y(x) &= \int_0^L g(x,a)F(a)\,da \\
&= \int_0^L g(x,a)K\,da \\
&= \int_0^x g\boxed{\text{seg 2}} \times K\,da + \int_x^L g\boxed{\text{seg 1}} \times K\,da \\
&= \int_0^x \frac{a(L-x)K}{TL}\,da + \int_x^L \frac{x(L-a)K}{TL}\,da \\
&= \left(\frac{K}{2T}\right) x(L-x)
\end{aligned}
$$

Section 4: Summary

If $f(t) = g(t) = 0$ for all $t < 0$ then

$$\int_0^t g(t-\tau)f(\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

Systems for which $g(t) = 0$ for all $t < 0$ are called *causal* systems.

Systems with time-varying inputs and outputs are causal.

Systems that have inputs and outputs that vary as a function of spatial location can have $g(x) \neq 0$ for any x.

We have learnt how to handle a spatially-varying impulse response.

Section 5

Fourier Series

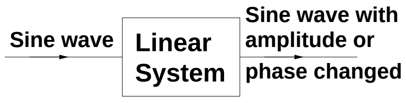
The Fourier series is introduced using an analogy with splitting vectors up into components.

The symmetry properties that enable us to predict that certain coefficients are zero are presented.

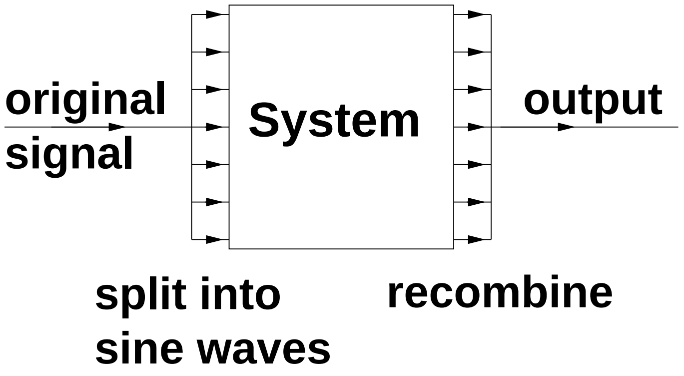
Motivation

We mentioned at the start of the last section that sine waves have a special property in relation to linear systems.

A sine wave at the input leads to a (possibly different) sine wave at the output.

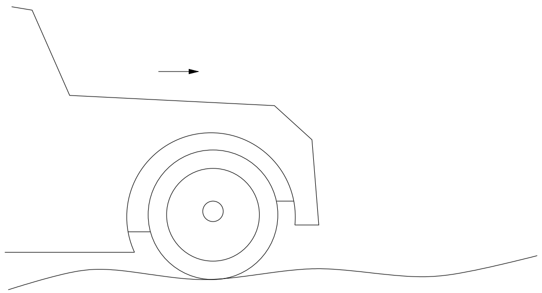


It would therefore be useful to be able to express an arbitrary signal in terms of a sum of sine waves.



Motivation: Car Suspension

Supposing we know that our car suspension will start to oscillate (bounce up and down uncomfortably) at frequency f.



We want to measure a variety of typical road profiles and calculate how much of frequency f they each contain (with the car travelling at a particular speed).

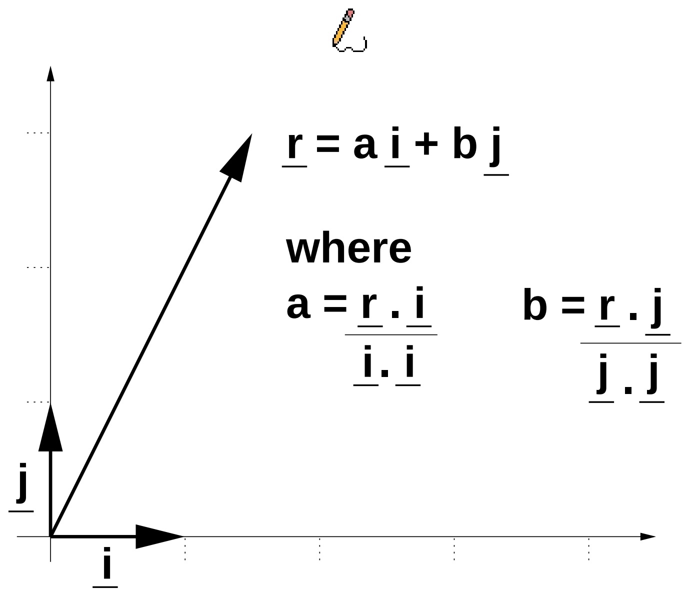
This will tell us which combinations of road profile and speed are likely to be a problem.

The Fourier series enables us to represent the road profile as the sum of a set of sinusoidal components at different frequencies.

Splitting up Vectors

We want to express a signal $f(t)$ in the range $-\pi \leq t \leq \pi$ in terms of some basic signals, i.e. sine waves. Let's look first at how we do a similar thing with vectors.

Consider how we express the arbitrary vector $\underline{r}$ in terms of the basis vectors $\underline{i}$ and $\underline{j}$.



The basis vectors are orthogonal: $\underline{i}.\underline{j} = 0$.

Basis Functions

Just as we represent $\underline{r}$ using orthogonal basis vectors, we want to represent $f(t)$ in the range $-\pi$ to π using orthogonal basis functions. We only need two vectors, but we need an infinite number of functions.

$$\begin{array}{lllll} \multicolumn{5}{l}{1 \text{ (i.e. a constant term)}} \\ \cos(t) & \cos(2t) & \cos(3t) & \cos(4t) & \ldots \\ \sin(t) & \sin(2t) & \sin(3t) & \sin(4t) & \ldots \end{array}$$

If n and m are positive integers greater than zero.

$$\begin{aligned}
\textstyle\int_{-\pi}^{\pi} \cos(nt)\sin(mt)\,dt &= 0 \\
\textstyle\int_{-\pi}^{\pi} \cos(nt) \times 1\,dt &= 0 \\
\textstyle\int_{-\pi}^{\pi} \sin(nt) \times 1\,dt &= 0 \\
\int_{-\pi}^{\pi} \cos(nt)\cos(mt)\,dt &= \begin{cases} 0 & , n \neq m \\ \pi & , n = m \end{cases} \\
\int_{-\pi}^{\pi} \sin(nt)\sin(mt)\,dt &= \begin{cases} 0 & , n \neq m \\ \pi & , n = m \end{cases} \\
\int_{-\pi}^{\pi} 1 \times 1\,dt &= 2\pi
\end{aligned}$$

So, using $\int_{-\pi}^{\pi} p(t)q(t)\,dt$ as our “dot product for functions”, the basis functions are orthogonal.

Fourier Series

The equivalents of our vector dot product expressions to calculate the component of $\underline{r}$ in each direction (eg. $a = (\underline{r}.\underline{i})/(\underline{i}.\underline{i})$) are:

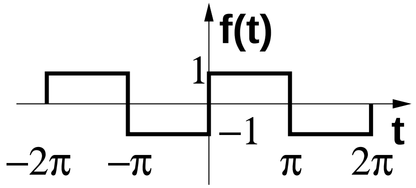
$$\begin{aligned}
a_n &= \frac{\int_{-\pi}^{\pi} \cos(nt) f(t)\,dt}{\int_{-\pi}^{\pi} \cos(nt)\cos(nt)\,dt} &&= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt) f(t)\,dt \\
b_n &= \frac{\int_{-\pi}^{\pi} \sin(nt) f(t)\,dt}{\int_{-\pi}^{\pi} \sin(nt)\sin(nt)\,dt} &&= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt) f(t)\,dt \\
d &= \frac{\int_{-\pi}^{\pi} 1 \times f(t)\,dt}{\int_{-\pi}^{\pi} 1 \times 1\,dt} &&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt
\end{aligned}$$

The equivalent of our vector expression for $\underline{r}$ in terms of $\underline{i}$ and $\underline{j}$, (i.e. $\underline{r} = a\underline{i} + b\underline{j}$) is an expression for f in terms of all the basis functions.

$$f(t) = \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) + d \times 1$$

Fourier Series Example 1

Represent the square wave $f(t)$ as a Fourier series.



$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt) f(t)\,dt &&= 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt) f(t)\,dt &&= \frac{2\,(1-(-1)^n)}{n\pi} \\
d &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt &&= 0
\end{aligned}
$$

Thus, we can model the square wave function $f(t)$ using:

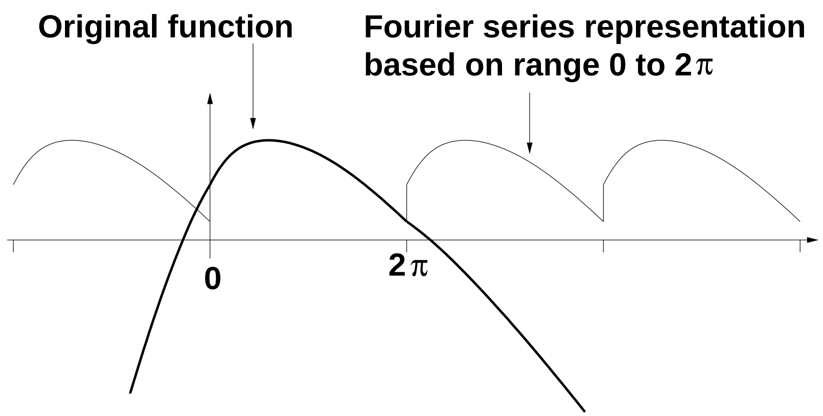
$$
\begin{aligned}
f(t) &= d + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right) \\
&= \sum_{n=1}^{\infty} \frac{2\,(1-(-1)^n)}{n\pi} \sin(nt) \\
&= \frac{4}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]
\end{aligned}
$$

Fourier Series Properties

1. We can use any range of length 2π instead of $-\pi \leq t \leq \pi$ in the Fourier formulae. For example, $0 \leq t \leq 2\pi$ is equally OK.

2. We are only modelling the function $f(t)$ in the specified range (eg. $-\pi$ to π, or 0 to 2π). Outside this range the model will just repeat with period 2π.

This is fine if the function we wish to model is periodic itself, but if the function is not periodic the Fourier model will probably only be useful over the range on which it was built.



Fourier Series Example 2

Represent $f(t) = e^t$ as a Fourier series between $-\pi$ and π.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt)e^t\,dt = \frac{(-1)^n\left(e^{\pi} - e^{-\pi}\right)}{\pi\left(1+n^2\right)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt)e^t\,dt = \frac{-(-1)^n\left(e^{\pi} - e^{-\pi}\right)n}{\pi\left(1+n^2\right)}$$

$$d = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^t\,dt = \frac{e^{\pi} - e^{-\pi}}{2\pi}$$

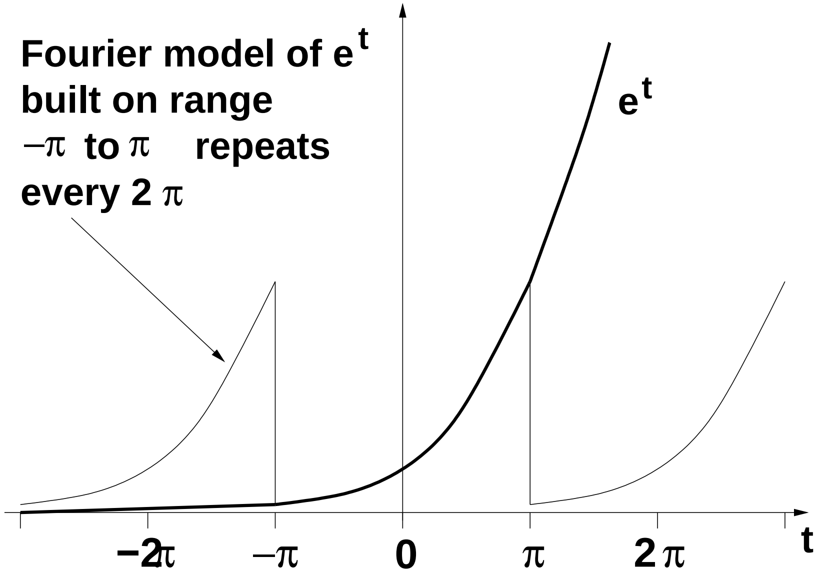
Thus, in the range $-\pi < t < \pi$ we can model the function $f(t) = e^t$ using:

$$f(t) = d + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

$$= \frac{e^{\pi} - e^{-\pi}}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\left[\cos(nt) - n\sin(nt)\right]\right)$$

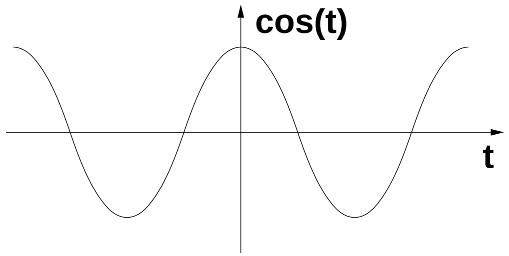
$$\approx 3.68 - 3.68\cos(t) + 3.68\sin(t) + 1.47\cos(2t) - 2.94\sin(2t) - \ldots$$

Fourier Model of Exponential

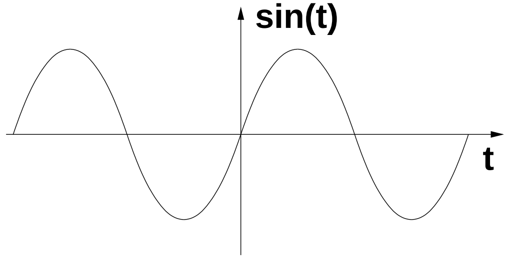


Symmetric Signals

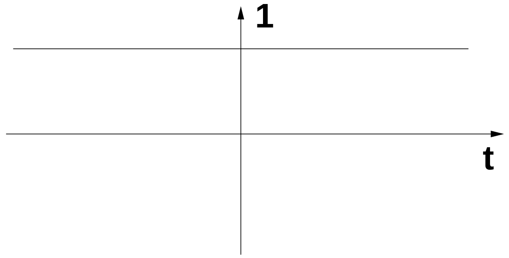
ODD function	$f(-t) = -f(t)$	eg: $\sin(t)$
EVEN function	$f(-t) = f(t)$	eg: $\cos(t)$



The a_n terms model the EVEN component in the function



The b_n terms model the ODD component in the function

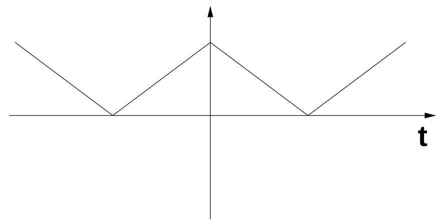


The d term models the mean value of the function

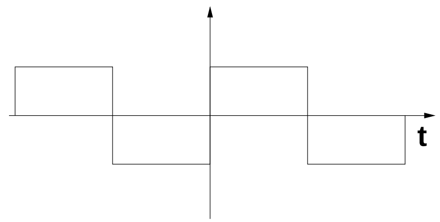
Avoiding Integration

If we can spot a symmetry in the function to be represented then we can avoid evaluating one or more of the Fourier integrals.

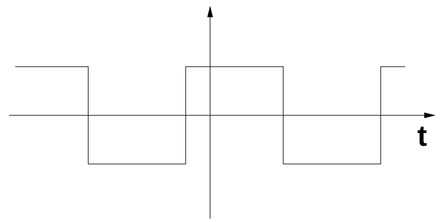
No even component $\Rightarrow$ all $a_n = 0$
No odd component $\Rightarrow$ all $b_n = 0$
Zero mean $\Rightarrow$ $d = 0$



EVEN function with non−zero mean: $b_n = 0$



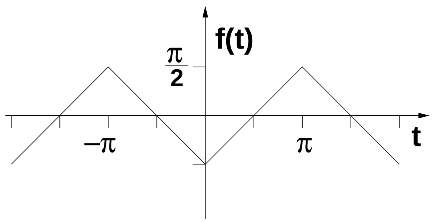
Purely ODD function with zero mean: $a_n = 0$ and $d = 0$



Function with zero mean: $d = 0$

Fourier Series Example 3

Find the Fourier series representation for the function $f(t)$ below.



EVEN function with zero mean: $b_n = 0$ and $d = 0$

We only have to calculate a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt) f(t)\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \cos(nt)(-t-\pi/2)\, dt + \frac{1}{\pi}\int_{0}^{\pi} \cos(nt)(t-\pi/2)\, dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \cos(nt)(t-\pi/2)\, dt$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & , n \text{ even} \\ \frac{-4}{n^2\pi} & , n \text{ odd} \end{cases}$$

so the Fourier series is:

$$f(t) = \frac{-4}{\pi}\left[\cos(t) + \frac{1}{9}\cos(3t) + \frac{1}{25}\cos(5t) + \ldots\right]$$

Fourier Series Example 4

Find the Fourier series representation for the function $f(t) = \cos(t + \pi/4)$.

This function has a mean value of zero so $d = 0$.

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt)\cos(t+\pi/4)\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nt + t + \frac{\pi}{4}) + \cos(nt - t - \frac{\pi}{4})\, dt \\ &= \frac{1}{\sqrt{2}}, \text{ when } n = 1 \text{ and 0 otherwise.} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt)\cos(t+\pi/4)\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nt + t + \frac{\pi}{4}) + \sin(nt - t - \frac{\pi}{4})\, dt \\ &= \frac{-1}{\sqrt{2}}, \text{ when } n = 1 \text{ and 0 otherwise.} \end{aligned}$$

so the Fourier series is:

$$f(t) = \frac{\cos(t) - \sin(t)}{\sqrt{2}}$$

Section 5: Summary

Periodic functions, (so far only with period 2π), can be represented using the the Fourier series.

We can use symmetry properties of the function to spot that certain Fourier coefficients will be zero, and hence avoid performing the integral to evaluate them.

- Functions with zero mean have $d = 0$.
- Purely odd functions have $a_n = 0$.
- Purely even functions have $b_n = 0$.

Segments of non-periodic functions can be represented using the Fourier series in the same way. The Fourier series representation just repeats outside the range on which it was built.

Section 6

General Fourier Series

The Fourier series for arbitrary period is presented.

We compare three techniques for calculating a general range Fourier series: direct integration, using a related series of delta functions, and using the maths data book.

During the direct integration example, some symmetry arguments for simplifying integrals are illustrated.

General Range

If we want to model a periodic signal with period other than 2π, or a section of a non-periodic signal of length other than 2π we need a more general formula.

To model a function $f(x)$ over the range 0 to L, substitute $\frac{2\pi x}{L} = t$, $\left(\Rightarrow \frac{2\pi}{L}dx = dt\right)$ in our Fourier formulae.

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L \cos\left(\frac{2\pi n x}{L}\right) f(x)\,dx \\
b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\,dx \\
d &= \frac{1}{L}\int_0^L f(x)\,dx
\end{aligned}
$$

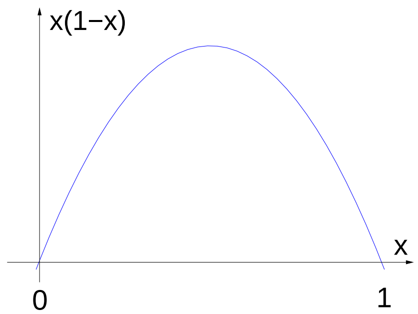
$$f(x) = d + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right]$$

The fraction $\frac{2\pi}{L}$ is often written as ω_0 and called the

fundamental angular frequency.

General Range Example 1

Represent the signal $f(x) = x(1-x)$ as a Fourier series with period 1, based on the range 0 to 1.



$$
\begin{aligned}
a_n &= 2\int_0^1 \cos(2\pi n x)x(1-x)\,dx = \frac{-1}{n^2\pi^2} \\
b_n &= 2\int_0^1 \sin(2\pi n x)x(1-x)\,dx = 0 \\
d &= \int_0^1 x(1-x)\,dx = \frac{1}{6}
\end{aligned}
$$

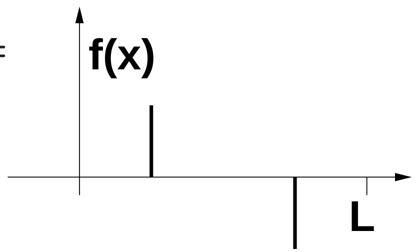
So the Fourier series is:

$$f(x) = \frac{1}{6} - \frac{\cos(2\pi x)}{\pi^2} - \frac{\cos(4\pi x)}{4\pi^2} - \frac{\cos(6\pi x)}{9\pi^2} - \ldots$$

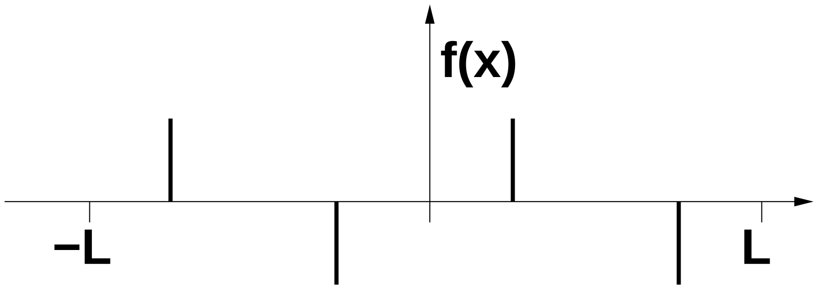
Note that this is an even function with period $= 1$.

General Range Example 2

Represent the signal $f(x) = \delta(x - L/4) - \delta(x - 3L/4)$ as a Fourier series based on the range 0 to L.



We are told that the period is L, so consider the signal repeating with period L.



This signal is purely ODD with zero mean. We therefore only need to calculate b_n.

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\,dx \\
&= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right)\left[\delta\left(x - \frac{L}{4}\right) - \delta\left(x - \frac{3L}{4}\right)\right]\,dx \\
&= \frac{2}{L}\left[\sin\left(\frac{2\pi n L}{4L}\right) - \sin\left(\frac{6\pi n L}{4L}\right)\right] \qquad \text{(sifting!)} \\
&= \frac{2}{L}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right] \\
&= \frac{4}{L}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

$$b_n = \frac{4}{L} \sin\left(\frac{n\pi}{2}\right)$$

This is zero when n is even. Tabulate $\sin\left(\frac{n\pi}{2}\right)$ when n is odd.

$\sin\left(\frac{n\pi}{2}\right)$	n	$\frac{n+1}{2}$	$-1^{\left(\frac{n+1}{2}\right)}$	$\frac{n+3}{2}$	$-1^{\left(\frac{n+3}{2}\right)}$
1	1	1	-1	2	1
-1	3	2	1	3	-1
1	5	3	-1	4	1
-1	7	4	1	5	-1

Thus

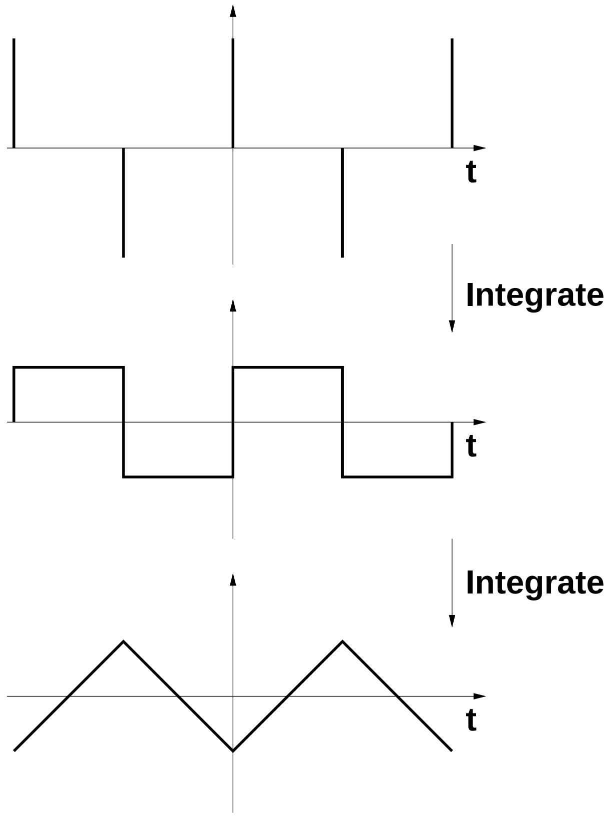
$$b_n = \begin{cases} 0 & , n \text{ even} \\ \frac{4}{L}(-1)^{\left(\frac{n+3}{2}\right)} & , n \text{ odd} \end{cases}$$

So the Fourier series is:

$$f(x) = \frac{4}{L}\left[\sin\left(\frac{2\pi x}{L}\right) - \sin\left(\frac{6\pi x}{L}\right) + \sin\left(\frac{10\pi x}{L}\right) - \ldots\right]$$

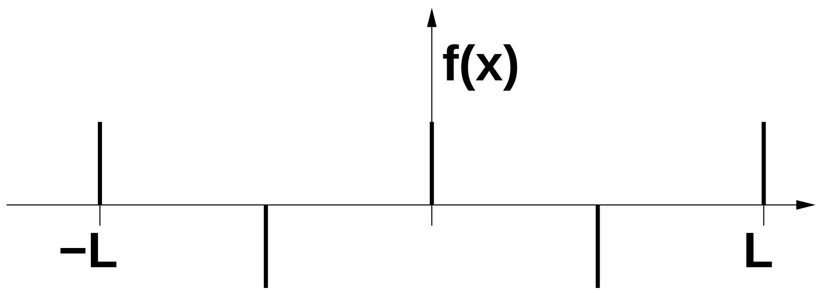
More Integral Avoidance

Notice how easy it is to calculate the Fourier series of a signal formed only of delta functions. By integrating the delta function series we can derive the Fourier series for square waves and triangle waves.



Pick the Start of Period Carefully

If you wish to find the Fourier series of a waveform such as



it is difficult to use formulae with limits such as

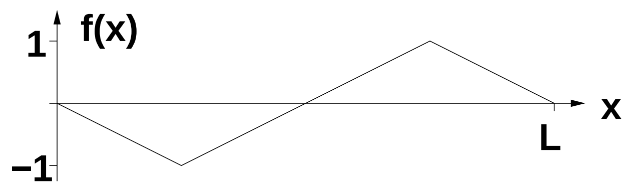
$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{2\pi nx}{L}\right) f(x)\,dx$$

because it is not clear what to do about the delta functions at that coincide with the upper and lower limits of the integral.

Instead, choose your period of length L to start at a different point. For example:

$$a_n = \frac{2}{L}\int_{\frac{-L}{4}}^{\frac{3L}{4}} \cos\left(\frac{2\pi nx}{L}\right) f(x)\,dx$$

Three Methods



There are three ways to find the Fourier series for $f(x)$ between 0 and L.

1. Use the general range Fourier formulae directly.

2. Differentiate the waveform twice to get a sequence of delta functions. Find a Fourier series for the delta functions, then integrate the series twice to get the Fourier series of the triangular wave.

3. Look up the Fourier series of a similar waveform in the Maths Data book and use a substitution of variables to find the series for the waveform we require.

Method 1: Direct Integration

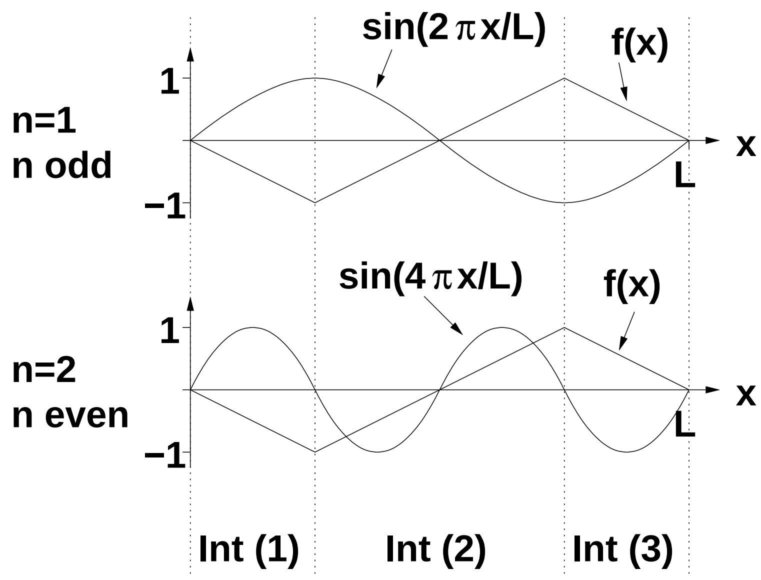
The triangular waveform is entirely ODD and has zero mean. Thus $d = 0$ and $a_n = 0$. We only need to find b_n.

To do this we need an algebraic representation of the waveform.

$$f(x) = \begin{cases} -\frac{4x}{L} & , 0 < x < \frac{L}{4} \\ \frac{4x}{L} - 2 & , \frac{L}{4} < x < \frac{3L}{4} \\ 4 - \frac{4x}{L} & , \frac{3L}{4} < x < L \end{cases}$$

From this we can write down an expression for b_n.

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\, dx \\ &= \frac{2}{L}\int_0^{\frac{L}{4}} \sin\left(\frac{2\pi n x}{L}\right)\left(\frac{-4x}{L}\right)\, dx & (1) \\ &\quad + \frac{2}{L}\int_{\frac{L}{4}}^{\frac{3L}{4}} \sin\left(\frac{2\pi n x}{L}\right)\left(\frac{4x}{L} - 2\right)\, dx & (2) \\ &\quad + \frac{2}{L}\int_{\frac{3L}{4}}^{L} \sin\left(\frac{2\pi n x}{L}\right)\left(4 - \frac{4x}{L}\right)\, dx & (3) \end{aligned}$$



There is clearly a symmetry between the terms $f(x)$ and $\sin\left(\frac{2\pi n x}{L}\right)$.

All terms with even n are zero, and all terms with odd n are equal to twice integral (2).

When n is even $b_n = 0$ and when n is odd

$$b_n = \frac{8}{n^2\pi^2}\left(\frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right)$$

But as we know n is odd, the $\cos()$ term is always zero and we can write $\left[-\sin\left(\frac{n\pi}{2}\right)\right] = (-1)^{\left(\frac{n+1}{2}\right)}$

$$\Rightarrow b_n = \begin{cases} 0, & n \text{ even} \\ \frac{8}{n^2\pi^2} \times (-1)^{\left(\frac{n+1}{2}\right)}, & n \text{ odd} \end{cases}$$

Giving a final Fourier series for $f(x) =$

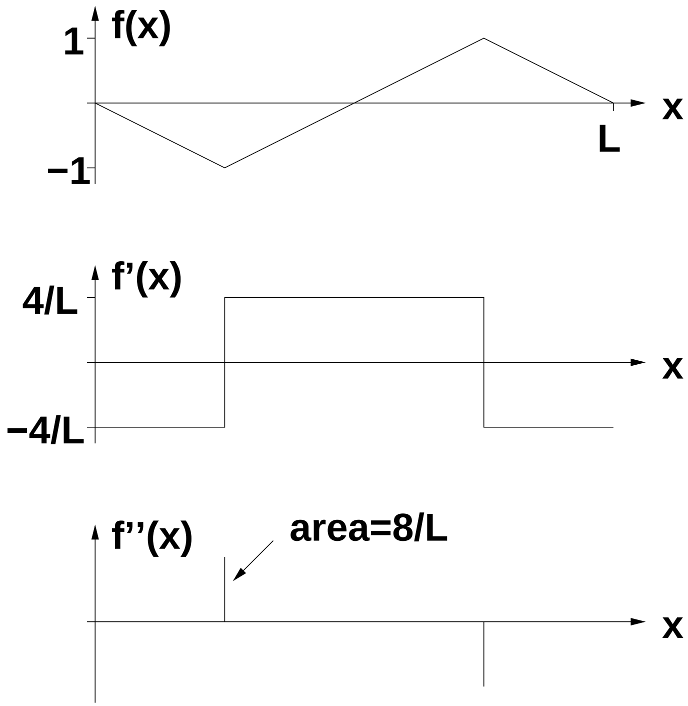
$$\frac{8}{\pi^2}\left[-\sin\left(\frac{2\pi x}{L}\right) + \frac{\sin\left(\frac{2\pi 3x}{L}\right)}{9} - \frac{\sin\left(\frac{2\pi 5x}{L}\right)}{25} + \dots\right]$$

If we want to write this algebraically, we need to limit n to only odd values. Let $n = 2m - 1$ with m taking integer values from 1 to ∞.

$$f(x) = \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)^2}\sin\left(\frac{2\pi x(2m-1)}{L}\right)$$

Method 2: Delta Functions

First we differentiate the waveform twice.



$f''(x)$ is a purely odd function with zero mean so we only need to calculate b_n.

$$f''(x) = \frac{8}{L}\delta\left(x - \frac{L}{4}\right) - \frac{8}{L}\delta\left(x - \frac{3L}{4}\right)$$

To find the Fourier series for $f''(x)$:

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\,dx \\ &= \tfrac{16}{L^2}\int_0^L \sin\left(\tfrac{2\pi n x}{L}\right)\left[\delta\left(x-\tfrac{L}{4}\right)-\delta\left(x-\tfrac{3L}{4}\right)\right]\,dx \\ &= \frac{16}{L^2}\left[\sin\left(\frac{2\pi n L}{4L}\right)-\sin\left(\frac{6\pi n L}{4L}\right)\right] \qquad \text{(sifting!)} \\ &= \begin{cases} 0 & , n \text{ even} \\ \tfrac{32}{L^2}(-1)^{\left(\frac{n+3}{2}\right)} & , n \text{ odd} \end{cases} \end{aligned}$$

So the Fourier series for $f''(x) =$

$$\frac{32}{L^2}\left[\sin\left(\frac{2\pi x}{L}\right)-\sin\left(\frac{6\pi x}{L}\right)+\sin\left(\frac{10\pi x}{L}\right)-\ldots\right]$$

We can also write this (note that $2m-1=n$).

$$f''(x) = \frac{32}{L^2}\sum_{m=1}^{\infty}(-1)^{m+1}\sin\left(\frac{2\pi x(2m-1)}{L}\right)$$

Now we integrate twice, each time setting the constant of integration to zero so we get a waveform with zero mean in each case.

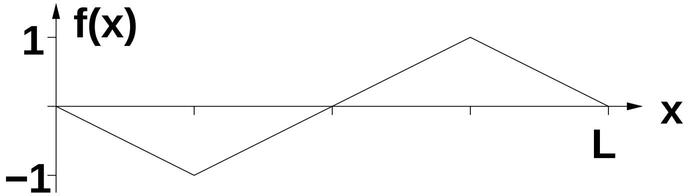
$$\begin{aligned} f''(x) &= \frac{32}{L^2}\sum_{m=1}^{\infty}\sin\left(\frac{2\pi x(2m-1)}{L}\right)(-1)^{m+1} \\ f'(x) &= \frac{16}{\pi L}\sum_{m=1}^{\infty}\frac{\cos\left(\frac{2\pi x(2m-1)}{L}\right)}{2m-1}(-1)^{m} \\ f(x) &= \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{2\pi x(2m-1)}{L}\right)}{(2m-1)^2}(-1)^{m} \end{aligned}$$

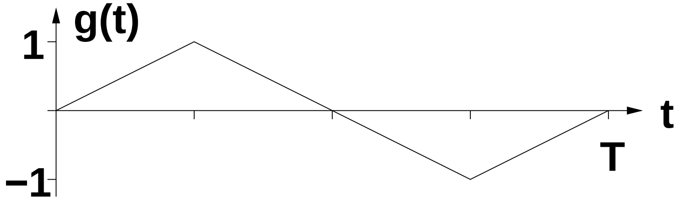
Which we can write out as follows $f(x) =$

$$\frac{8}{\pi^2}\left[-\sin\left(\frac{2\pi x}{L}\right)+\frac{\sin\left(\frac{2\pi 3x}{L}\right)}{9}-\frac{\sin\left(\frac{2\pi 5x}{L}\right)}{25}+\ldots\right]$$

Method 3: Maths Databook

Only works if something like the desired function is in the maths data book!





In this case we want $f(x)$ as above, and the nearest available series is $g(t)$.

$$g(t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin([2n-1]\omega_0 t)}{(2n-1)^2}$$

where $\omega_0 = 2\pi/T$.

If we set $x = t$ and $L = T$ then $f = -g$.

$$\begin{aligned} \Rightarrow f(x) &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin([2n-1]\omega_0 x)}{(2n-1)^2} (-1)^n \\ &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2\pi x(2n-1)}{L}\right)}{(2n-1)^2} (-1)^n \end{aligned}$$

Which we can write out, as with the other methods, as follows $f(x) =$

$$\frac{8}{\pi^2}\left[-\sin\left(\frac{2\pi x}{L}\right) + \frac{\sin\left(\frac{2\pi 3x}{L}\right)}{9} - \frac{\sin\left(\frac{2\pi 5x}{L}\right)}{25} + \ldots\right]$$

Section 6: Summary

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L \cos\left(\frac{2\pi n x}{L}\right) f(x)\,dx \\
b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\,dx \\
d &= \frac{1}{L}\int_0^L f(x)\,dx \\
f(x) &= d + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right]
\end{aligned}
$$

You can sometimes combine multiple integrals using symmetry properties.

Sometimes it is faster to calculate a related Fourier series of delta functions and integrate.

Don't forget the Fourier serieses given in the maths data book.

Section 7

Convergence & Half Range Serieses

The rule for predicting the convergence of the Fourier series from the shape of the function is introduced.

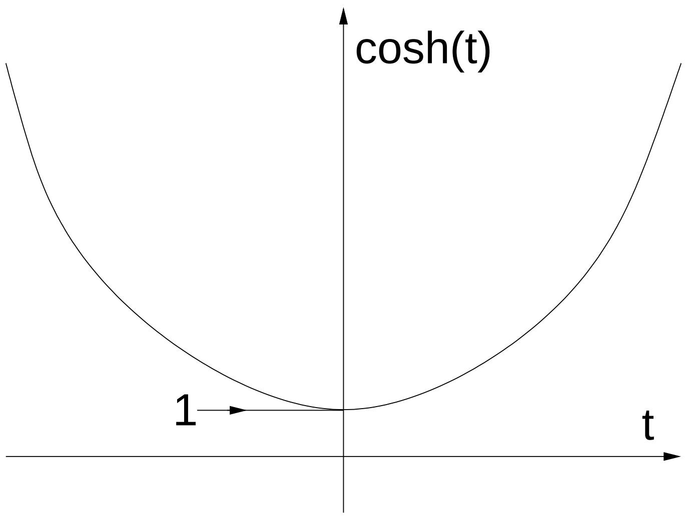
This is used with the Fourier series for general period to calculate serieses, valid over limited ranges, with improved convergence properties. Four different serieses are calculated to model the same simple function in order to illustrate this.

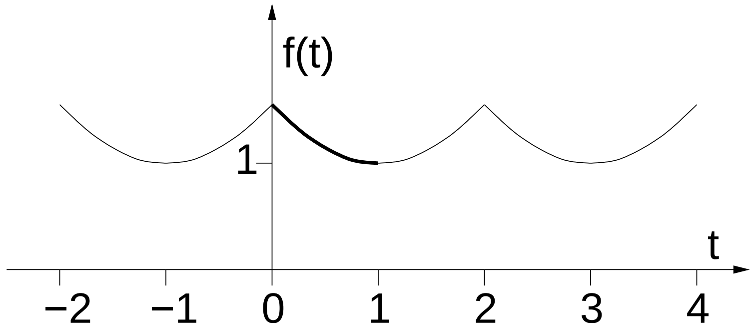
The usefulness of Matlab and Octave for numerical calculation, and the use of Matlab for symbolic algebra are introduced.

General Range Example 3

A even function $f(t)$ is periodic with period $T = 2$, and $f(t) = \cosh(t-1)$ for $0 \le t \le 1$. Sketch $f(t)$ in the range $-2 \le t \le 4$. Find a Fourier series representation for $f(t)$.

First remember what the graph of $\cosh(t)$ looks like.





It is an even function $\Rightarrow b_n = 0$ and $a_n \neq 0$. The mean value of the function is non-zero $\Rightarrow d \neq 0$.

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_{-1}^{1} f(t)\cos\left(\frac{2\pi n t}{T}\right)\,dt \\
&= \frac{4}{T}\int_{0}^{1} f(t)\cos\left(\frac{2\pi n t}{T}\right)\,dt \\
&= 2\int_{0}^{1} \cosh(t-1)\cos(n\pi t)\,dt \\
&= \frac{2\sinh(1)}{1+n^2\pi^2}
\end{aligned}
$$

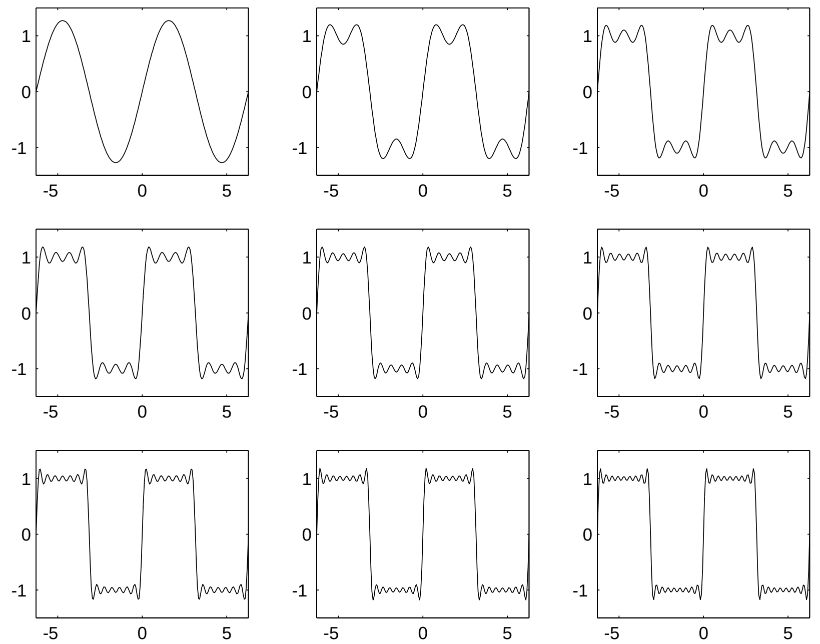
$$
\begin{aligned}
d &= \frac{1}{T}\int_{-1}^{1} f(t)\,dt = \frac{2}{T}\int_{0}^{1} f(t)\,dt \\
&= \int_{0}^{1} \cosh(t-1)\,dt = \sinh(1)
\end{aligned}
$$

So

$$
\begin{aligned}
f(t) &= d + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right) \\
&= \sinh(1)\left[1 + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi t)}{1+n^2\pi^2}\right]
\end{aligned}
$$

Square Wave Series Convergence

The graphs below show the sum of 1, 2, 3 . . . up to 9 terms of the Fourier series for a square wave.



Using Matlab/Octave

```
% Fourier series for square wave
number = 200;
dtheta = 4*pi/number;
theta = -2*pi:dtheta:2*pi;
nharm = 20;
d = 0;
thing = d * ones(1,number+1);
for n=1:nharm
  if mod(n,2) == 1
    bn = 4/(pi*n);
  else
    bn = 0;
  end
  an = 0;
  thing = thing + an * cos(n*theta) ...
              + bn * sin(n*theta);
  plot(theta,thing);
  axis([-2*pi 2*pi -1.5 1.5]);
  pause(1)
end
```



```
theta = -2*pi:dtheta:2*pi;
```

sets up theta as an array with 201 elements, starting at -2π, going up to 2π, with spacing `dtheta`$=4\pi/200$.

$$-6.2832, -6.2204, \ldots \qquad \ldots 6.2204, 6.2832$$

```
thing = d * ones(1,number+1);
```

initialises the 201 element array in which we hold the value of the series at each angle. The initial value of each element is d, which in this case is zero.

```
for n=1:nharm
```

This introduces a `for` loop. We go round the loop `nharm` times to add in `nharm` harmonics.

```
thing = thing + an * cos(n*theta) ...
              + bn * sin(n*theta);
```

This statement works on every element of the `theta` array, calculating the terms of the cos and sin serieses and adding them in to the appropriate sums in the `thing` array.

Using Matlab Symbolic Tools

Both convolution and Fourier work involves a lot of integration. Sometimes it is nice to know what the right answer is, so you can check your working. To integrate p with respect to x from a to b you use the command `int(p, x, a, b)`. Consider the integral:

$$\frac{2}{T}\int_0^T \cos\left(\frac{2\pi n x}{T}\right) x\, dx = 0$$

```
>> syms n x T
>> int((2/T)*x*cos(2*pi*n*x/T),x,0,T)

ans =
T*(cos(pi*n)^2-1
 +2*pi*n*sin(pi*n)*cos(pi*n))/pi^2/n^2
```

which is

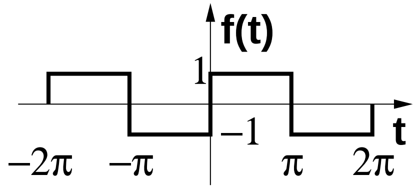
$$\frac{T\left((\cos(\pi\, n))^2 - 1 + 2\,\pi\, n \sin(\pi\, n)\cos(\pi\, n)\right)}{\pi^2 n^2}$$

But as n is an integer, $\cos^2(n\pi) = 1$ and $\sin(n\pi) = 0$, so the integral evaluates to zero.

Don't rely on this too much. You need to be able to integrate efficiently by hand in the exam.

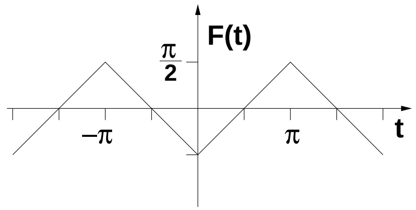
Convergence Examples

The Fourier series for a square wave converges as $1/n$. Notice that it is discontinuous of value.



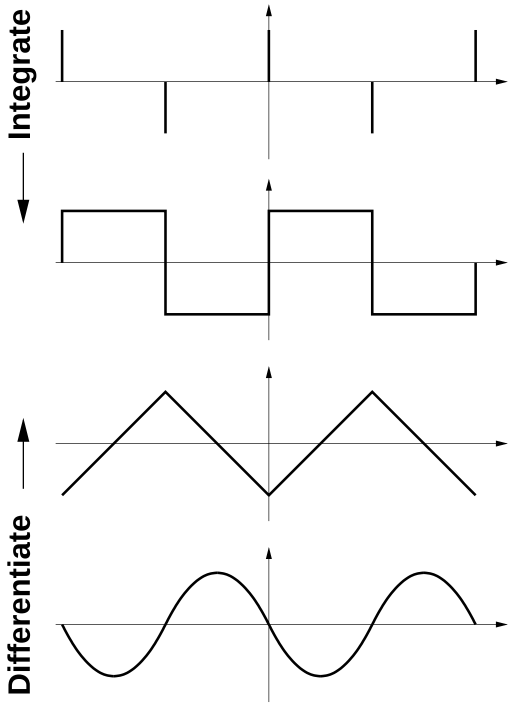
$$f(t) = \frac{4}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$$

The Fourier series for a triangular wave converges as $1/n^2$. It is continuous of value, but discontinuous of gradient.



$$F(t) = \frac{-4}{\pi}\left[\cos(t) + \frac{\cos(3t)}{9} + \frac{\cos(5t)}{25} + \ldots\right]$$

Convergence



Series for delta functions does not converge

Discontinuous value: converges as 1/n

Discontinuous gradient: converges as $1/n^2$

Discontinuous second derivative: converges as $1/n^3$

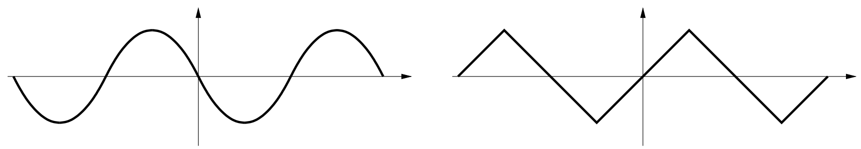
Odd Numbers

If $m = 1, 2, 3, 4, 5, 6, 7 \ldots$

and $n = 2m - 1$ and $m = \dfrac{n+1}{2}$

then $n = 1, 3, 5, 7, 9, 11, 13 \ldots$

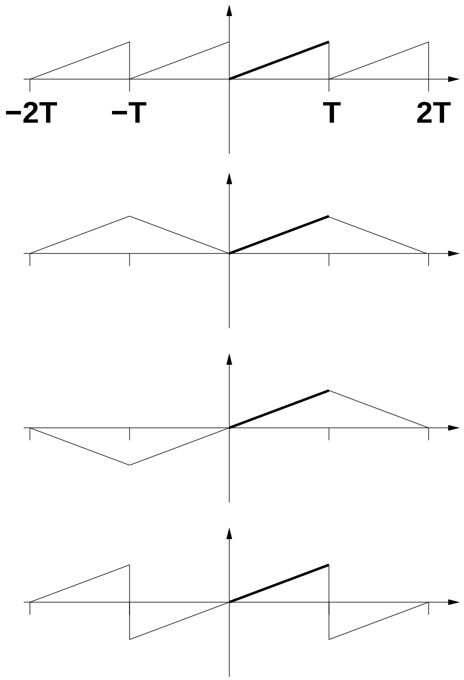
Odd Functions



$$f(x) = -f(-x)$$

“Half Range” Series

If we want to model a signal $f(x) = x$ in the range 0 to T. We can use the Fourier formulae for general range to generate a variety of different serieses. They will all be the same in the range 0 to T, but some may converge faster than others.



Full range series
period T
converges as 1/n

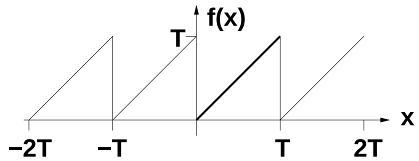
Cosine series
period 2T, $b_n = 0$
converges as $1/n^2$

Sine series
period 4T, $a_n = 0$, d = 0
converges as $1/n^2$

Sine series
period 2T, $a_n = 0$, d = 0
converges as 1/n

Normal Series, Period T

Find the Fourier series to model $f(x) = x$ from 0 to T, using a series of period T.

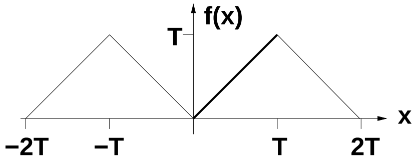


$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^T \cos\left(\frac{2\pi n x}{T}\right) x\,dx = 0 \\
b_n &= \frac{2}{T}\int_0^T \sin\left(\frac{2\pi n x}{T}\right) x\,dx = \frac{-T}{n\pi} \\
d &= \frac{1}{T}\int_0^T x\,dx = \frac{T}{2} \\
\Rightarrow f(x) &= \frac{T}{2} - \sum_{n=1}^{\infty} \frac{T}{n\pi}\sin\left(\frac{2\pi n x}{T}\right)
\end{aligned}
$$

Notice that the series converges as $1/n$.

Cosine Series, Period 2T

Find the Fourier series to model $f(x) = x$ from 0 to T, using a cosine series of period $2T$.



$$a_n = \frac{1}{T}\int_{-T}^{T} \cos\left(\frac{\pi n x}{T}\right) f(x)\, dx$$

$$= \frac{1}{T}\left[\int_{-T}^{0} \cos\left(\frac{\pi n x}{T}\right)(-x)\, dx + \int_{0}^{T} \cos\left(\frac{\pi n x}{T}\right) x\, dx\right]$$

$$= \frac{2}{T}\int_{0}^{T} \cos\left(\frac{\pi n x}{T}\right) x\, dx = \frac{-4T}{n^2\pi^2}, \text{ only } n \text{ ODD}$$

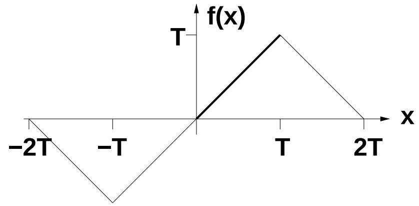
$$d = \frac{1}{2T}\int_{-T}^{T} f(x)\, dx = \frac{T}{2}$$

$$f(x) = \frac{T}{2} - \sum_{m=1}^{\infty} \frac{4T}{(2m-1)^2\pi^2} \cos\left(\frac{(2m-1)\pi x}{T}\right)$$

Notice that the series converges as $1/n^2$.

Sine Series, Period 4T

Find the Fourier series to model $f(x) = x$ from 0 to T, using a sine series of period $4T$.



Notice how the function is symmetrical about T (i.e. $\frac{1}{4}$ of the period). This leads to $b_n = 0$ when n is even because all such terms are anti-symmetric about T.

$$b_n = \frac{1}{2T}\int_{-2T}^{2T} \sin\left(\frac{\pi n x}{2T}\right) f(x)\, dx$$

$$= \frac{1}{T}\int_{-T}^{T} \sin\left(\frac{\pi n x}{2T}\right) x\, dx, \; n \text{ odd only}$$

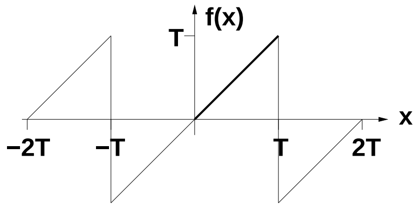
$$= \frac{8T(-1)^{\left(\frac{n+3}{2}\right)}}{n^2\pi^2}, \; n \text{ odd only}$$

$$f(x) = \sum_{m=1}^{\infty} \frac{8T(-1)^{(m+1)}}{(2m-1)^2\pi^2} \sin\left(\frac{\pi(2m-1)x}{2T}\right)$$

Notice that the series converges as $1/n^2$.

Sine Series, Period 2T

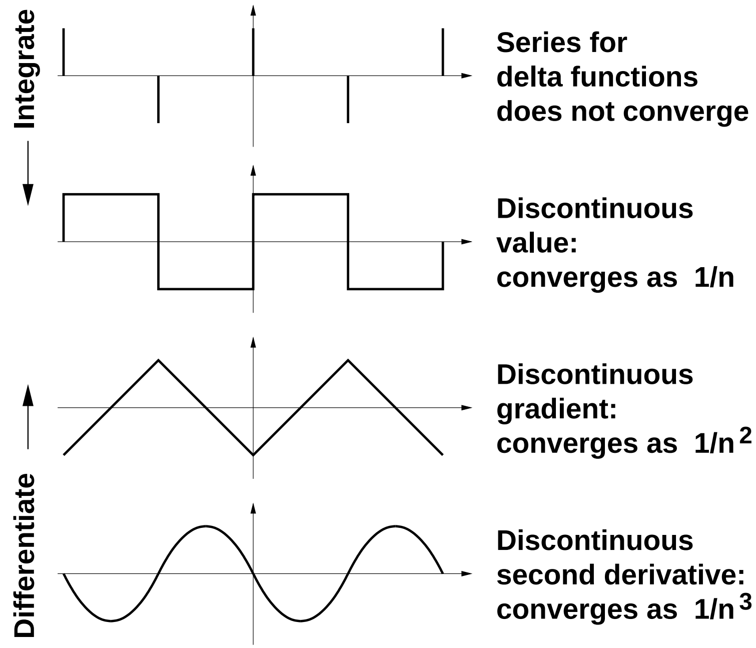
Find the Fourier series to model $f(x) = x$ from 0 to T, using a sine series of period $2T$.



$$
\begin{aligned}
b_n &= \frac{1}{T}\int_{-T}^{T} \sin\left(\frac{\pi n x}{T}\right) x\,dx \\
&= \frac{-2T}{n\pi}(-1)^n \\
\Rightarrow f(x) &= \sum_{n=1}^{\infty} \frac{-2T}{n\pi}(-1)^n \sin\left(\frac{\pi n x}{T}\right)
\end{aligned}
$$

Notice that the series converges as $1/n$.

Section 7: Summary



If you are modelling a limited section of a function, pick the Fourier series period so as to get good convergence and a series that is easy to calculate (i.e. some of a_n, b_n or d zero).

Section 8

Complex Fourier Series

The complex Fourier series is presented first with period 2π, then with general period.

The connection with the real-valued Fourier series is explained and formulae are given for converting between the two types of representation.

Examples are given of computing the complex Fourier series and converting between complex and real serieses.

New Basis Functions

Recall that the Fourier series builds a representation composed of a weighted sum of the following basis functions.

$$\begin{array}{lllll} \multicolumn{5}{l}{1 \text{ (i.e. a constant term)}} \\ \cos(t) & \cos(2t) & \cos(3t) & \cos(4t) & \ldots \\ \sin(t) & \sin(2t) & \sin(3t) & \sin(4t) & \ldots \end{array}$$

Computing the weights a_n, b_n and d often involves some nasty integration.

We now present an alternative representation based on a different set of basis functions:

$$\begin{array}{lllll} \multicolumn{5}{l}{1 \text{ (i.e. a constant term)}} \\ e^{it} & e^{2it} & e^{3it} & e^{4it} & \ldots \\ e^{-it} & e^{-2it} & e^{-3it} & e^{-4it} & \ldots \end{array}$$

These can all be represented by the term

$$e^{int}$$

with n taking integer values from $-\infty$ to $+\infty$. Note that the constant term is provided by the case when $n = 0$.

Series of Complex Exponentials

A representation based on this family of functions is called the "complex Fourier series".

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

The coefficients, c_n, are normally complex numbers.

It is often easier to calculate than the sin/cos Fourier series because integrals with exponentials in are usually easy to evaluate.

We will now derive the complex Fourier series equations, as shown above, from the sin/cos Fourier series using the expressions for sin() and cos() in terms of complex exponentials.

Complex Fourier Series

$$f(t) = d + \sum_{n=1}^{\infty} \left[a_n \cos(nt) + b_n \sin(nt)\right]$$

$$= d + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{int} + e^{-int}}{2}\right) + b_n \left(\frac{e^{int} - e^{-int}}{2i}\right)\right]$$

$$= d + \sum_{n=1}^{\infty} \frac{(a_n - ib_n)}{2} e^{int} + \sum_{n=1}^{\infty} \frac{(a_n + ib_n)}{2} e^{-int}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{int}$$

where

$$c_n = \begin{cases} d & , n = 0 \\ (a_n - ib_n)/2 & , n = 1, 2, 3, \ldots \\ (a_{-n} + ib_{-n})/2 & , n = -1, -2, -3, \ldots \end{cases}$$

Note that a_{-n} and b_{-n} are only defined when n is negative.

$$
\begin{aligned}
a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt) f(t)\,dt \\
b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt) f(t)\,dt \\
d &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt
\end{aligned}
$$

thus for n positive

$$
\begin{aligned}
c_n &= \frac{1}{2}(a_n - ib_n) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos(nt) - i\sin(nt)]\, f(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt
\end{aligned}
$$

for n negative

$$
\begin{aligned}
c_n &= \frac{1}{2}(a_{-n} + ib_{-n}) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos(-nt) + i\sin(-nt)]\, f(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt
\end{aligned}
$$

and for $n = 0$

$$
\begin{aligned}
c_0 &= d \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-0} f(t)\,dt
\end{aligned}
$$

Complex Fourier Series Summary

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt \\
f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{int}
\end{aligned}
$$

Complex Series Example 1

Find the complex Fourier series to model $f(t) = \sin(t)$.

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\sin(t)\,dt \\
&= \frac{1}{2\pi}\left[\frac{e^{in\pi}-e^{-in\pi}}{n^2-1}\right]
\end{aligned}
$$

Which is zero when n does not equal 1 or -1. For these two special cases we have to set $n = 1+\epsilon$ and calculate the limit of c_n as ϵ tends to zero. This gives us

$$
\begin{aligned}
c_1 &= \frac{1}{2i} \\
c_{-1} &= \frac{-1}{2i}
\end{aligned}
$$

Which means the complex Fourier series for $f(t) = \sin(t)$ is

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{int} \\
&= \frac{e^{it}-e^{-it}}{2i}
\end{aligned}
$$

Finding the limit as n tends to 1

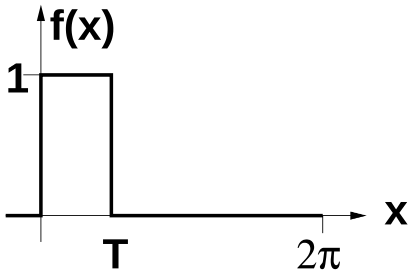
$$
c_n = \frac{1}{2\pi}\left[\frac{e^{in\pi}-e^{-in\pi}}{n^2-1}\right]
$$

Set $n = 1+\epsilon$ and let ϵ tend to zero.

$$
\begin{aligned}
c_1 &= \frac{1}{2\pi}\left[\frac{e^{i\pi(1+\epsilon)}-e^{-i\pi(1+\epsilon)}}{(1+\epsilon)^2-1}\right] \\
&= \frac{1}{2\pi}\left[\frac{-e^{i\pi\epsilon}+e^{-i\pi\epsilon}}{(1+\epsilon)^2-1}\right] \\
&\approx \frac{1}{2\pi}\left[\frac{-1-i\pi\epsilon+1-i\pi\epsilon}{1+2\epsilon-1}\right] \\
&\approx \frac{1}{2\pi}\left[\frac{-2i\pi\epsilon}{2\epsilon}\right] \\
&\approx \frac{-i}{2} \\
&\approx \frac{1}{2i}
\end{aligned}
$$

Complex Series Example 2

Find the complex Fourier series to model $f(x)$ that has a period of 2π and is 1 when $0 < x < T$ and zero when $T < x < 2\pi$.



$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)\,dt \\
&= \frac{i}{2\pi n}\left[e^{-inT}-1\right] \text{, when } n \neq 0 \\
&= \frac{1}{2\pi}\text{area} = \frac{T}{2\pi} \text{, when } n = 0
\end{aligned}
$$

So the Fourier series is

$$
f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}
$$

$$
= \frac{1}{2\pi}\left\{T + \sum_{n=-\infty}^{-1} \frac{i}{n}\left[e^{-inT}-1\right]e^{int} + \sum_{n=1}^{\infty} \frac{i}{n}\left[e^{-inT}-1\right]e^{int}\right\}
$$

Converting c to a, b and d

From our example on the previous page.

$$
c_n = \begin{cases} \frac{i}{2\pi n}\left[e^{-inT}-1\right] & \text{, when } n \neq 0 \\ \frac{1}{2\pi}\text{area} = \frac{T}{2\pi} & \text{, when } n = 0 \end{cases}
$$

We wish to calculate the coefficients for the equivalent Fourier series in terms of $\sin()$ and $\cos()$.

Clearly $d = c_0 = \frac{T}{2\pi}$. For $n > 0$

$$
\begin{aligned}
c_n &= (a_n - ib_n)/2 \\
\Rightarrow a_n &= 2\,\textbf{Re}\{c_n\} \\
\text{and } b_n &= -2\,\textbf{Im}\{c_n\}
\end{aligned}
$$

converting our expression for c_n into sin() and cos():

$$
\begin{aligned}
2c_n &= \frac{i}{\pi n}\left[\cos(nT) - i\sin(nT) - 1\right] \\
&= \frac{1}{\pi n}\left[\sin(nT) + i(\cos(nT) - 1)\right]
\end{aligned}
$$

so $a_n = \dfrac{\sin(nT)}{n\pi}$ and $b_n = \dfrac{1-\cos(nT)}{n\pi}$.

Complex Fourier Series

$$f(t) = \frac{1}{2\pi}\left\{T + \sum_{n=-\infty}^{-1} \frac{i}{n}\left[e^{-inT} - 1\right] e^{int} \right.$$
$$\left. + \sum_{n=1}^{\infty} \frac{i}{n}\left[e^{-inT} - 1\right] e^{int}\right\}$$

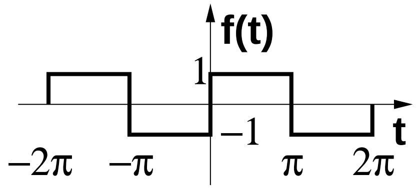
Real Fourier Series

$$f(t) = \frac{T}{2\pi} + \sum_{n=1}^{\infty} \frac{\sin(nT)}{n\pi}\cos(nt)$$
$$+ \sum_{n=1}^{\infty} \frac{1-\cos(nT)}{n\pi}\sin(nt)$$

Both serieses converge as $1/n$.

Converting from Real to Complex

Convert the real Fourier series of the square wave $f(t)$ to a complex series.



For the real series, we know that $d = a_n = 0$ and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt) f(t)\, dt = \frac{4}{n\pi}, \; n \text{ odd}$$

giving $f(t) = \frac{4}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$

To convert to a complex series, use

$$c_n = \begin{cases} d & , n = 0 \\ (a_n - ib_n)/2 & , n = 1, 2, 3, \ldots \\ (a_{-n} + ib_{-n})/2 & , n = -1, -2, -3, \ldots \end{cases}$$

so we have

$$\begin{aligned} c_0 &= 0 \\ c_n &= -2i/(n\pi) & , n \text{ positive and odd} \\ c_n &= 2i/(-n\pi) & , n \text{ negative and } |n| \text{ odd} \end{aligned}$$

$$\Rightarrow f(t) = \frac{-2i}{\pi}\left[\ldots + \frac{e^{-5it}}{-5} + \frac{e^{-3it}}{-3} + \frac{e^{-it}}{-1} \right.$$
$$\left. + \frac{e^{it}}{1} + \frac{e^{3it}}{3} + \frac{e^{5it}}{5} + \ldots\right]$$

General Complex Series

For period of 2π

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)\,dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

Similarly, for period L

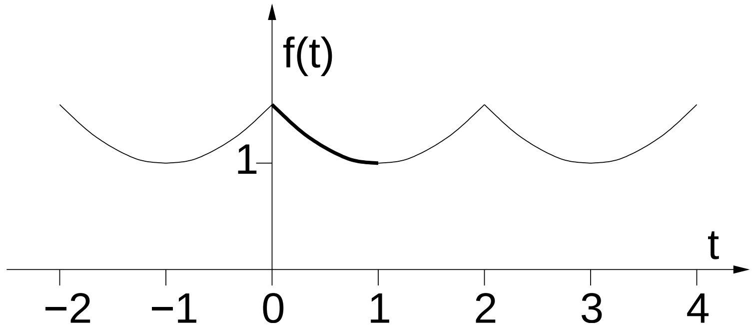
$$c_n = \frac{1}{L}\int_0^{L} e^{-inx\frac{2\pi}{L}} f(x)\,dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx\frac{2\pi}{L}}$$

The fraction $\frac{2\pi}{L}$ is often written as ω_0 and called the fundamental angular frequency.

Example 1

A even function $f(t)$ is periodic with period $L = 2$, and $f(t) = \cosh(t-1)$ for $0 \le t \le 1$. Find a complex Fourier series representation for $f(t)$.



$$\begin{aligned} c_n &= \frac{1}{L}\int_0^{L} e^{-int\frac{2\pi}{L}} f(t)\,dt \\ &= \frac{1}{2}\int_0^{2} e^{-int\pi} \cosh(t-1)\,dt \\ &= \frac{\sinh(1)}{1+n^2\pi^2} \end{aligned}$$

Hence the complex Fourier series is

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{int\frac{2\pi}{L}} \\
&= \sum_{n=-\infty}^{\infty} \frac{\sinh(1) e^{int\pi}}{1+n^2\pi^2}
\end{aligned}
$$

We can check this answer by computing the equivalent real Fourier series which we calculated at the start of section 7.

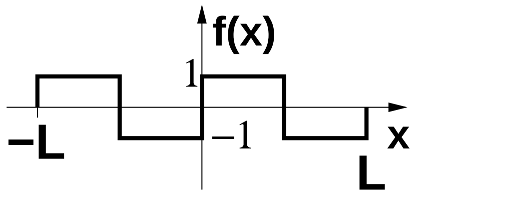
$$
\begin{aligned}
a_n &= 2\,\textbf{Re}\{c_n\} \quad , n = 1, 2, 3, \ldots \\
b_n &= -2\,\textbf{Im}\{c_n\} \quad , n = 1, 2, 3, \ldots \\
d &= c_0
\end{aligned}
$$

In this case, as c_n is entirely real,

$$
\begin{aligned}
a_n &= 2c_n = \frac{2\sinh(1)}{1+n^2\pi^2} \; , n = 1, 2, 3, \ldots \\
b_n &= 0 \\
d &= \sinh(1)
\end{aligned}
$$

Example 2

Find the complex Fourier series of the the square wave $f(x)$.



Note that the mean of the function is zero, so $c_0 = 0$.

$$
\begin{aligned}
c_n &= \frac{1}{L}\int_0^L e^{-inx\frac{2\pi}{L}} f(x)\,dx \\
&= \frac{1}{L}\left[\int_0^{L/2} e^{-inx\frac{2\pi}{L}}\,dx - \int_{L/2}^{L} e^{-inx\frac{2\pi}{L}}\,dx\right] \\
&= \frac{1}{2in\pi}\left[e^{-2in\pi} + 1 - 2e^{-in\pi}\right]
\end{aligned}
$$

$$
f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\left[1 - e^{-in\pi}\right]}{in\pi} e^{inx\frac{2\pi}{L}}
$$

$$
\begin{aligned}
f(x) = \frac{2}{i\pi}\Bigg[\ldots &+ \frac{e^{-5ix\frac{2\pi}{L}}}{-5} + \frac{e^{-3ix\frac{2\pi}{L}}}{-3} + \frac{e^{-ix\frac{2\pi}{L}}}{-1} \\
&+ \frac{e^{ix\frac{2\pi}{L}}}{1} + \frac{e^{3ix\frac{2\pi}{L}}}{3} + \frac{e^{5ix\frac{2\pi}{L}}}{5} + \ldots\Bigg]
\end{aligned}
$$

Converting to a Real Series

We wish to convert the complex general range square wave series into a series with real coefficients.

$$c_n = \begin{cases} 2/(in\pi) & , |n| \text{ odd} \\ 0 & , |n| \text{ even} \end{cases}$$

Clearly $d = c_0 = 0$. For a and b use:

$$\begin{aligned} c_n &= (a_n - ib_n)/2 \\ \Rightarrow a_n &= 2\,\mathbf{Re}\{c_n\} = 0 \\ \text{and } b_n &= -2\,\mathbf{Im}\{c_n\} = \frac{4}{n\pi}, \; n \text{ odd} \end{aligned}$$

Which gives us the real series:

$$f(t) = \frac{4}{\pi}\left[\sin\left(x\frac{2\pi}{L}\right) + \frac{\sin\left(3x\frac{2\pi}{L}\right)}{3} + \frac{\sin\left(5x\frac{2\pi}{L}\right)}{5} + \ldots\right]$$

Section 8: Summary

For period L

$$\begin{aligned} c_n &= \frac{1}{L}\int_0^L e^{-inx\frac{2\pi}{L}} f(x)\,dx \\ f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx\frac{2\pi}{L}} \end{aligned}$$

Relationship with the cos/sin Fourier series.

$$c_n = \begin{cases} d & , n = 0 \\ (a_n - ib_n)/2 & , n = 1, 2, 3, \ldots \\ (a_{-n} + ib_{-n})/2 & , n = -1, -2, -3, \ldots \end{cases}$$

$$\begin{aligned} a_n &= 2\,\mathbf{Re}\{c_n\} & , n = 1, 2, 3, \ldots \\ b_n &= -2\,\mathbf{Im}\{c_n\} & , n = 1, 2, 3, \ldots \\ d &= c_0 \end{aligned}$$

Section 9

Probability

In this section we summarise the key issues in pages 1–13 of the basic probability teach-yourself document and provide a single simple example of each concept.

This presentation is intended to be reinforced by the many examples in the teach-yourself document and the first 12 questions of examples paper 10.

Probability

Probability of A $=$

$$\frac{\text{Number of outcomes for which } A \text{ happens}}{\text{Total number of outcomes (sample space)}}$$

What is the probability of drawing an ace from a shuffled pack of cards?

There are 4 aces. There are 52 cards in total. Therefore the probability is

$$\mathbf{P}(\text{ ace }) = \frac{4}{52} = \frac{1}{13}$$

Adding Probabilities

$$\mathbf{P}(A \text{ or } B) = \mathbf{P}(A) + \mathbf{P}(B)$$

provided A and B cannot happen together, i.e. A and B must be mutually *exclusive* outcomes.

What is the probability of drawing an ace or a king from a shuffled pack of cards?

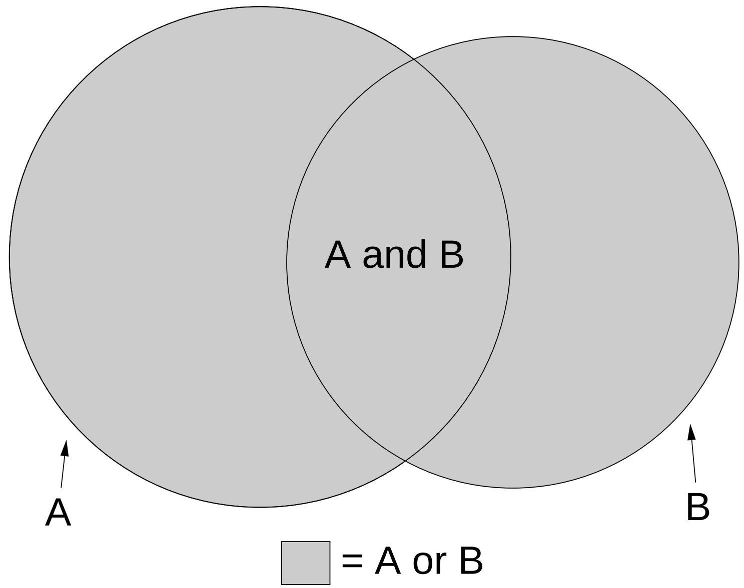
$$\mathbf{P}(\text{ ace }) = \frac{1}{13}$$

$$\mathbf{P}(\text{ king }) = \frac{1}{13}$$

$$\Rightarrow \mathbf{P}(\text{ ace or king }) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

When Not to Add Probabilities

When the events are not mutually exclusive.



$$\mathbf{P}(A \text{ or } B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \text{ and } B)$$

Non-Exclusive Events

What is the probability of drawing an ace or a spade from a shuffled pack of cards?

$$\mathbf{P}(\text{ ace }) = \frac{1}{13} \qquad \mathbf{P}(\text{ spade }) = \frac{13}{52} = \frac{1}{4}$$

but **P**(ace or spade) is not the sum of these values because the outcomes "ace" and "spade" are not exclusive; it is possible to have them both together by drawing the ace of spades.

To calculate **P**(ace or spade)

either use the formula from the previous slide:

$$\mathbf{P}(\text{ ace }) + \mathbf{P}(\text{ spade }) - \mathbf{P}(\text{ ace of spades })$$
$$\frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{4}{13}$$

or use the original definition of probability.

$$\frac{\text{number of aces and spades}}{\text{total number of cards}} = \frac{4+13-1}{52} = \frac{4}{13}$$

Multiplying Probabilities

$$\mathbf{P}(A \text{ and } B) = \mathbf{P}(A) \times \mathbf{P}(B)$$

provided A is not affected by the outcome of B and B is not affected by the outcome of A, i.e. A and B must be *independent*.

I have two shuffled packs of cards and draw a card from each of them. What is the probability that I draw two aces?

$$\mathbf{P}(\text{ ace }) = \frac{1}{13}$$

$$\mathbf{P}(\text{ ace and ace }) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$$

Non-independent Events

I have a single pack of cards. I draw a card, then draw a second card without putting the first card back in the pack. What is the probability that I draw two aces?

This time the probability that I get an ace as the second card is affected by whether or not I removed an ace from the pack when I drew the first card.

We use the notation $\mathbf{P}(B|A)$ to denote the probability that B happens, given that we know that A happened. This is called a conditional probability.

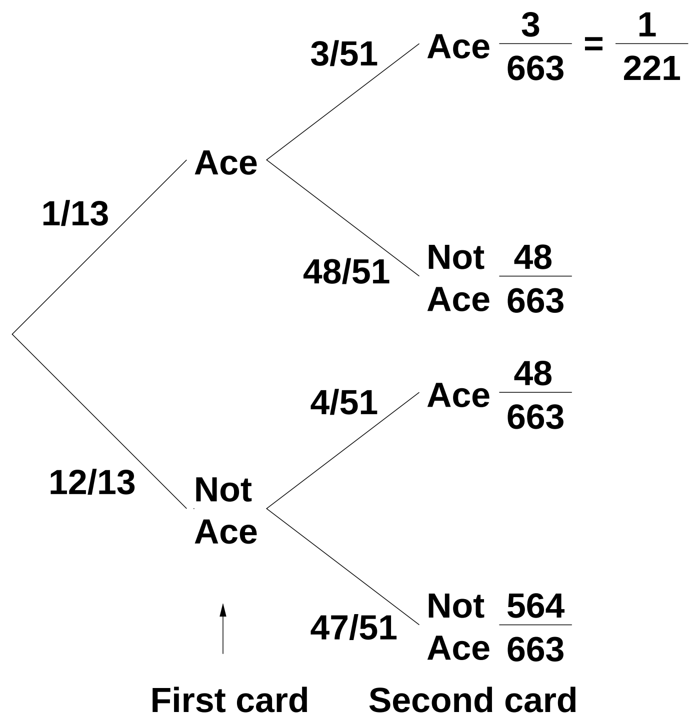
$$\mathbf{P}(A \text{ and } B) = \mathbf{P}(A|B)\,\mathbf{P}(B)$$

Thus:

$$\begin{aligned}&\mathbf{P}(\text{ [second = ace] and [first = ace] })\\ &\quad= \mathbf{P}(\text{ second = ace } | \text{ first = ace })\,\mathbf{P}(\text{ first = ace })\end{aligned}$$

Tree Diagram

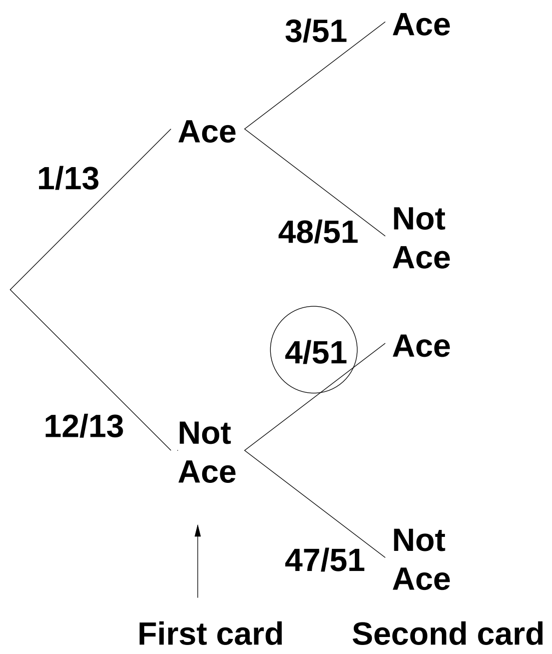
We can use a tree diagram to help us work this out.



The probability that both cards are aces $= \frac{1}{221}$.

Another Example

I have a single pack of cards. I draw a card, then draw a second card without putting the first card back in the pack. What is the probability that the second card is an ace, given that the first card was not an ace?



$$\mathbf{P}(\text{ second = ace} \mid \text{first = not ace }) = \frac{4}{51}$$

Notation

Intersection	$\bigcap$	AND
Union	$\bigcup$	OR

Thus the conditional probability formula

$$\mathbf{P}(A \text{ and } B) = \mathbf{P}(A|B)\,\mathbf{P}(B)$$

is more normally written

$$\mathbf{P}(A \cap B) = \mathbf{P}(A|B)\,\mathbf{P}(B)$$

and instead of

$$\mathbf{P}(A \text{ or } B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \text{ and } B)$$

we write

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$

Ordering Objects

The number of different orders in which n unique objects can be placed is $n!$ (n factorial)

I have three cards with values 2, 3 and 4. They are shuffled into a random order. What is the probability they are in the order 2, 3, 4?

The number of possible orders for three cards is $3!$ The probability the cards are found in one specific order is therefore $\frac{1}{3!} = \frac{1}{6}$.

Permutations

$$_nP_r = \frac{n!}{(n-r)!}$$

is the number of ways of choosing r items from n when the order of the chosen items matters.

Ten people are involved in a race. I wish to make a poster for every possible winning combination of gold, silver and bronze medal winners. How many posters will I need?

We need to know the number of ways of choosing three people out of 10, taking account of the order. This is

$$_{10}P_3 = \frac{10!}{7!} = 10 \times 9 \times 8 = 720$$

So I would need rather a lot of posters.

Combinations

$$_nC_r = \frac{n!}{(n-r)!r!}$$

is the number of ways of choosing r items from n when the order of the chosen items does not matter.

I have a single pack of cards. I draw a card, then draw a second card without putting the first card back in the pack. What is the probability that I draw two aces?

The number of ways of drawing 2 cards from 52 is $_{52}C_2$.

The number of ways of getting two aces is the number of ways of drawing 2 aces from the 4 aces in the pack. This is $_4C_2$.

The probability that I draw two aces is therefore

$$\frac{\text{num ace pairs}}{\text{num pairs}} = \frac{_4C_2}{_{52}C_2} = \frac{4!}{2!\,2!} \times \frac{50!\,2!}{52!}$$

$$= \frac{4 \times 3}{52 \times 51} = \frac{1}{221}$$

Lottery Example 1

What is the probability of winning the jackpot in the national lottery? There are 49 balls and you have to match all six to win.

Method 1:

$$\frac{6}{49} \times \frac{5}{48} \times \frac{4}{47} \times \frac{3}{46} \times \frac{2}{45} \times \frac{1}{44} = \frac{1}{13983816}$$

Method 2:

$$\frac{1}{\text{number of ways of choosing 6 balls from 49, where order does not matter}} = \frac{1}{_{49}C_6}$$

$$\frac{1}{_{49}C_6} = \frac{6!\,43!}{49!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{49 \times 48 \times 47 \times 46 \times 45 \times 44}$$

$$= \frac{1}{13983816}$$

Lottery Example 2

What is the probability of winning £10 by matching exactly 3 balls in the national lottery

Method 1:

Work out the probability of matching them in a particular order: the first 3 balls that are drawn win, the remaining 3 do not.

$$\frac{6}{49} \times \frac{5}{48} \times \frac{4}{47} \times \frac{43}{46} \times \frac{42}{45} \times \frac{41}{44}$$

Then multiply this by the number of possible ways of picking the 3 winning balls among the 6 balls that are drawn.

$$\left.\begin{array}{llllll} \checkmark & \checkmark & \checkmark & \times & \times & \times \\ \checkmark & \checkmark & \times & \checkmark & \times & \times \\ \checkmark & \checkmark & \times & \times & \checkmark & \times \\ \checkmark & \checkmark & \times & \times & \times & \checkmark \\ \checkmark & \times & \checkmark & \checkmark & \times & \times \\ \multicolumn{6}{l}{\ldots\text{ etc.}} \end{array}\right\} {}_6C_3 = 20$$

Hence: $\frac{6}{49} \times \frac{5}{48} \times \frac{4}{47} \times \frac{43}{46} \times \frac{42}{45} \times \frac{41}{44} \times {}_6C_3 = \frac{1}{56.7}$

Matching only 3 balls, method 2:

$$\frac{\text{number of ways we can win}}{\text{total possible number of outcomes}}$$

Thinking about all the balls in the lottery machine, we consider:

$$\frac{\begin{pmatrix}\text{The number of ways the} \\ \text{lottery machine can pick} \\ \text{3 balls matching some} \\ \text{of the 6 numbers on our} \\ \text{ticket.}\end{pmatrix} \begin{pmatrix}\text{The number of ways the} \\ \text{lottery machine can pick} \\ \text{3 balls from the 43 balls} \\ \text{not on our ticket.}\end{pmatrix}}{\begin{pmatrix}\text{The total number of ways of picking 6} \\ \text{balls out of the 49 in the machine.}\end{pmatrix}}$$

$$= \frac{{}_6C_3\,{}_{43}C_3}{{}_{49}C_6} = \frac{6!\,43!\,43!\,6!}{49!\,40!\,3!\,3!\,3!}$$

$$= \frac{43 \times 42 \times 41 \times 6 \times 5 \times 4 \times 6 \times 5 \times 4}{49 \times 48 \times 47 \times 46 \times 45 \times 44 \times 3 \times 2}$$

$$= \frac{1}{56.7}$$

Section 9: Summary

$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$ if A and B are mutually exclusive outcomes.

$\mathbf{P}(A \cap B) = \mathbf{P}(A) \times \mathbf{P}(B)$ provided A and B are independent.

$\mathbf{P}(A \cap B) = \mathbf{P}(A|B)\,\mathbf{P}(B)$

$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$

The number of different orders in which n unique objects can be placed is $n!$

Permutations: ${}_nP_r = \frac{n!}{(n-r)!}$ is the number of ways of choosing r items from n when the order of the chosen items matters.

Combinations: ${}_nC_r = \frac{n!}{(n-r)!r!}$ is the number of ways of choosing r items from n when the order of the chosen items does not matter.

Section 10

Statistics

In this section we summarise the key issues in pages 14–20 of the basic probability teach-yourself document. This presentation is intended to be reinforced by the examples in the teach-yourself document and questions 13 and 14 in examples paper 9.

The main focus is on the mean and standard deviation of a probability distribution. We also explain how to calculate a range within which we are (say) 95% sure that the true value of an experimental reading will lie.

Mean

The mean μ, of a population of values x_i (where i goes from 1 to N), is defined.

$$\mu = \frac{\text{Sum of all the values}}{\text{Number of values}} = \frac{\sum_{i=1}^{N} x_i}{N}$$

Imagine a pack of cards with all the jokers and picture cards removed. We are only concerned with the numerical value of the cards. We have four each of all the numbers from one to ten so $N = 40$.

$$\text{Arithmetic mean: } \mu = \frac{\sum_{i=1}^{N} x_i}{N} = \frac{220}{40} = 5.5$$

The mean is a measure of the *central tendency* or *location* of the population.

Variance & Standard Deviation

Variance and standard deviation are measures of the *spread* of the distribution. The variance is the average squared difference between each value and the mean. The population variance is usually given the symbol σ^2.

$$\text{Variance: } \sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}$$

The standard deviation (SD) is the square root of the variance. The population standard deviation is usually given the symbol σ.

$$\text{Standard Deviation: } \sigma = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}}$$

We can work out the variance and standard deviation of the values on our set of forty cards.

$$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N} = \frac{330}{40} = 8.25$$

$$\sigma = \sqrt{\frac{330}{40}} = 2.8723$$

Sample

Until now we have assumed that we can see all the cards at once. Now we are going to change the game. Imagine that someone else is holding the cards and allowing us to pick one at random, note its value and then replace it. Using this pick-and-replace process we can view a *sample* of the cards. This sample can be of any size as the cards are picked at random and replaced. Assume that the sample size is n.

The challenge is to estimate the mean and standard deviation of the original numbers on the cards based only on what we see in the sample. Here are the formulae that enable us to do this.

Estimate of Mean (based on sample):

$$m = \frac{\sum_{i=1}^{n} x_i}{n}$$

Estimate of Standard Deviation (based on sample):

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - m)^2}{n-1}}$$

Convenient Formula for s

In the literature, s, the standard deviation of the underlying population estimated from a sample is called the *“sample standard deviation”*.

There is a convenient formula for calculating s.

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - m)^2}{n-1}}$$

$$= \sqrt{\frac{\left(\sum_{i=1}^{n} x_i^2\right) - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}{n-1}}$$

Standard Deviation from a Sample

Ten cards are selected individually from our special reduced pack of 40 cards (described on slide 147), noted and replaced in the pack. This gives a sample size $n = 10$. The values of the cards are:

10	3	4	3	5
4	1	5	8	5

We wish to estimate the mean m, and standard deviation s, of the values on all the cards, based only on knowledge of this sample.

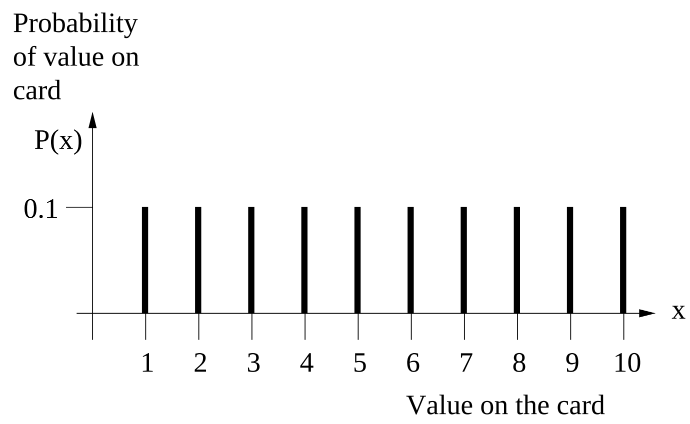
$$m = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{48}{10} = 4.8$$

$$s = \sqrt{\frac{\left(\sum_{i=1}^{n} x_i^2\right) - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}{n-1}}$$

$$= \sqrt{\frac{(290) - \frac{1}{10}(48)^2}{9}} = 2.5734$$

Discrete Probability Distribution

Consider picking a card (from the pack described on slide 147), noting its value and then replacing it in the pack. We can compute the probability of picking each of the possible values.



This is a probability distribution. In this case it is a *discrete* distribution because the cards can only carry certain integer values. Notice that the sum of all the histogram bars is $10 \times 0.1 = 1$. There are ten possible outcomes and they each have a probability of $1/10$. This is called a *uniform* distribution.

Mean and SD from the Distribution

The probability distribution is a property of the population of the numbers on the cards. Knowing the complete probability distribution enables us to calculate the mean μ and the standard deviation σ exactly.

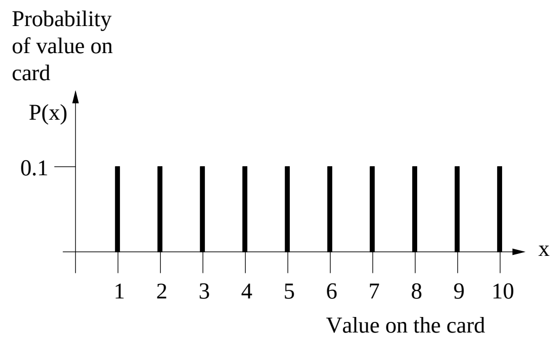
Let x_j represent each of the *different* values that are printed on the cards and M equal the number of these different values. In our example $M = 10$.

$$\text{Arithmetic mean: } \mu = \sum_{j=1}^{M} x_j P(x_j)$$

$$\text{Variance: } \sigma^2 = \sum_{j=1}^{M} (x_j - \mu)^2 P(x_j)$$

$$\text{Standard Deviation: } \sigma = \sqrt{\sum_{j=1}^{M} (x_j - \mu)^2 P(x_j)}$$

Example



We can see from the histogram that $P(x_j) = 0.1$ for all the values on the cards (i.e. for all j). In this particular case, the values x_j are the same numerically as the index j, so we can substitute $x_j = j$. Hence

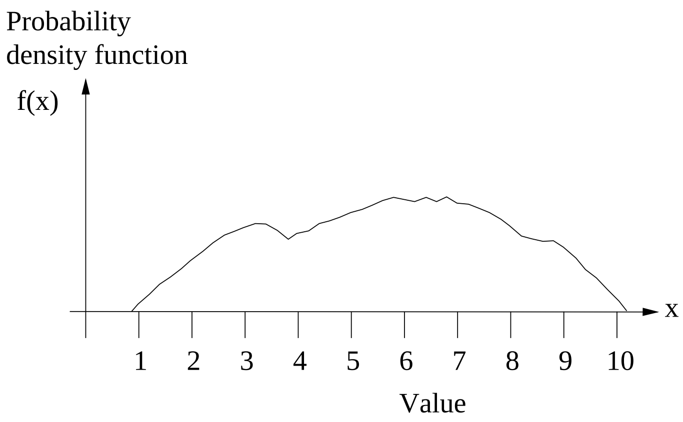
$$\mu = \sum_{j=1}^{M} x_j P(x_j) = \sum_{j=1}^{10} j \times 0.1 = 5.5$$

Now we use this value of μ in the formula for standard deviation.

$$\sigma = \sqrt{\sum_{j=1}^{M} (x_j - \mu)^2 P(x_j)}$$
$$= \sqrt{\sum_{j=1}^{10} (j - 5.5)^2 \times 0.1} = 2.8723$$

Continuous Probability Distribution

If you have an outcome that can take any real value (rather than a finite number of discrete values) this can be described by a probability density function (PDF).



Here the total area under the curve must be 1 and the probability of x taking a value in the range from (say) 6 to 7 is given by the integral (i.e. area) between 6 and 7. More generally:

$$\text{The probability of } (a < x < b) = \int_a^b f(x)\, dx$$

Examples of continuous random variables: the weight of a sample, the time for a physical process to complete, an output voltage.

Mean and SD from PDF

It is also possible to calculate the mean (μ) and standard deviation (σ) of a distribution from its probability density function.

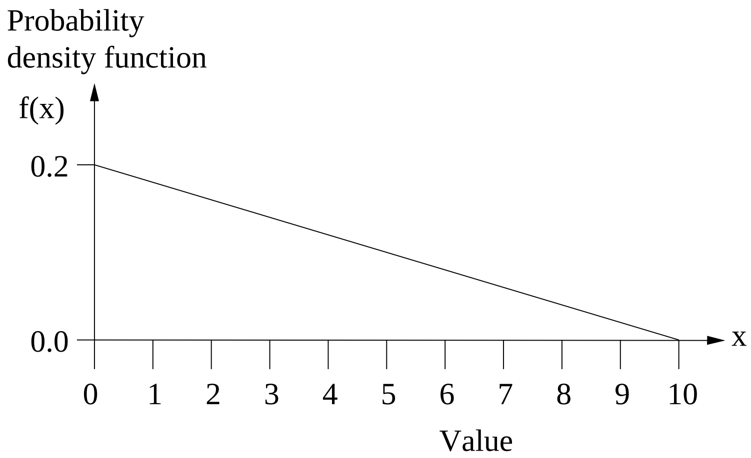
$$\mu = \int_{-\infty}^{+\infty} x f(x)\, dx$$

$$\sigma = \sqrt{\int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\, dx}$$

Knowing the probability density function enables us to calculate the mean μ and the standard deviation σ exactly.

Example using a PDF

Consider a machine than makes widgets which are supposed to be a particular length. Unfortunately, the machine often makes widgets that are slightly too long; it never makes widgets that are too short. The graph below shows the probability density function for the number of millimetres that a widget is too long.



Widget Example 1

1. What is the probability that a widget is less than 1 mm too long?

For $0 \leq x \leq 10$ we can see that $f(x) = 0.2 - 0.02x$, hence:

$$\mathrm{P}(0 < x < 1) = \int_0^1 f(x)\,dx$$

$$= \int_0^1 0.2 - 0.02x\,dx = 0.19$$

So the probability of a widget being less than 1 mm too long is 0.19.

2. Calculate the mean and standard deviation of the distribution of excess lengths.

$$\mu = \int_0^{10} x(0.2 - 0.02x)\,dx = 3.3333$$

$$\sigma = \sqrt{\int_0^{10} (x - 10/3)^2 (0.2 - 0.02x)\,dx} = 2.3570$$

Widget Example 2

3. What is the probability that a widget is produced with an excess length within one standard deviation from the mean excess length?

We wish to calculate the probability of an excess length in the range $3.33 - 2.36$ to $3.33 + 2.36$, which is given by:

$$\begin{aligned} \text{P}(x \text{ within one } \sigma \text{ of } \mu) &= \int_{0.9763}^{5.6904} 0.2 - 0.02x\,dx \\ &= 0.6285 \end{aligned}$$

4. The manufacturer wants to quote an excess length that he is sure 95% of the widgets produced will be shorter than. What should it be?

We need to solve for d in:

$$0.95 = \int_0^d 0.2 - 0.02x\,dx$$

$$\Rightarrow 0.95 = 0.2d - 0.01d^2$$

So $d = 10 - \sqrt{5} = 7.7639$ mm

Standard Deviation of Sample Mean

Suppose x is a random variable with

mean $= \mu$

standard deviation $= \sigma$

We have n samples of x: $[x_1, x_2, x_3, \ldots x_n]$.

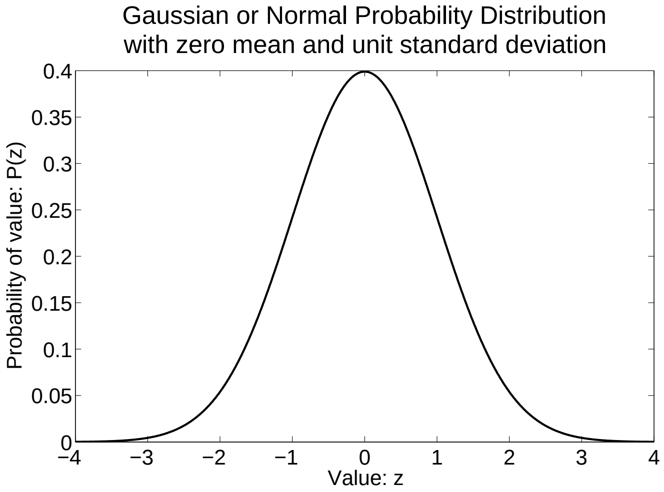
Let $\overline{x}$ be the average of these samples $= \sum_{i=1}^{n} \frac{x_i}{n}$.

Now the key point is that $\overline{x}$ is also a random variable.

The mean of $\overline{x}$ is μ.

The standard deviation of $\overline{x}$ is $\frac{\sigma}{\sqrt{n}}$.

Normal Probability Distribution



The Normal distribution is a symmetric distribution with two parameters, its mean μ and standard deviation σ.

$$P(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right)$$

If you add together a sufficient number (say 30), of independent random variables that are identically distributed, the sum will conform to the Normal distribution. This is called the “central limit theorem”.

Normal Distribution Example

Consider a laboratory experiment that results in a single real output x, each time that we perform it. In theory, it should produce the same output each time but in practice x varies slightly because of noise in the measurement system.

If we repeat the experiment at least 30 times and average the result ($\overline{x} = m = \sum_{i=1}^{n} x_i/n$) then we can say the following things about the way that $\overline{x}$ is distributed.

- Provided $n > 30$ it is reasonable to assume that $\overline{x}$ is Normally distributed.
- The standard deviation of $\overline{x}$ will be a factor of $\sqrt{n}$ less than the standard deviation of the original experimental data. Hence:

Estimate of standard deviation of $\overline{x}$:

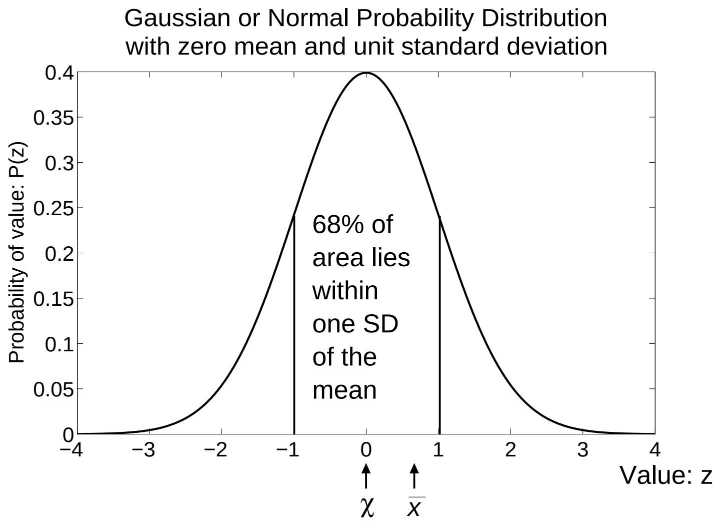
$$s_{(\overline{x})} = \frac{1}{\sqrt{n}} \times \sqrt{\frac{\left(\sum_{i=1}^{n} x_i^2\right) - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}{n-1}}$$

Normal Distribution Example 2

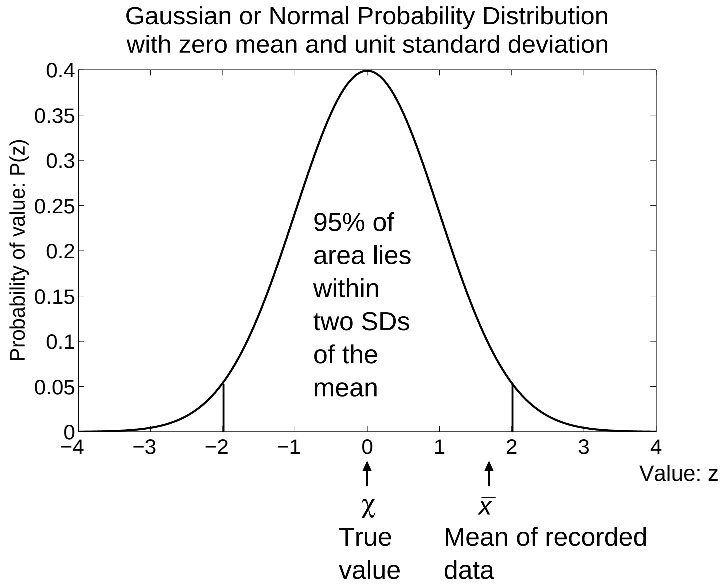
If it is fair to assume that the error in the original experimental data is unbiased, then the standard deviation of $\overline{x}$ gives us useful information about the error it is likely to contain.

Let $s_{(\overline{x})}$ be our estimate of the standard deviation of $\overline{x}$ and let $\mathcal{X}$ be the (unknown) true value of the thing we are intending to measure.

50%	of the time, $\overline{x}$ will lie within	$0.67s_{(\overline{x})}$	of $\mathcal{X}$
68%	of the time, $\overline{x}$ will lie within	$s_{(\overline{x})}$	of $\mathcal{X}$
95%	of the time, $\overline{x}$ will lie within	$2s_{(\overline{x})}$	of $\mathcal{X}$
99.73%	of the time, $\overline{x}$ will lie within	$3s_{(\overline{x})}$	of $\mathcal{X}$



Normal Distribution Example 3



If we repeat an experiment 30 times and

- the average result $\overline{x} = 3.0279$,
- we calculate an estimate of the standard deviation of the experimental error as 0.2036
- then our estimate of the standard deviation of $\overline{x}$ will be $0.2036/\sqrt{30} = 0.0372$.

The true experimental result will have a 95% chance of lying within two standard deviations from the mean, i.e. in the range from 2.9536 to 3.1023.

Section 10: Summary

PDF	probability density function
SD	standard deviation
μ	mean of a distribution (or from a PDF)
σ^2	variance of a distribution (or from a PDF)
σ	SD of a distribution (or from a PDF)
$m = \overline{x}$	estimate of μ based on a sample of x values
s	estimate of σ based on a sample of x values
$s_{(\overline{x})}$	estimate of SD of $\overline{x}$ based on sample of x $s_{(\overline{x})} = s/\sqrt{n}$ where n is the sample size.

If we are prepared to do an experiment at least 30 times and believe our results to be unbiased, we can use the "central limit theorem" together with the shape of the Normal distribution to calculate a range within which the true result is likely to lie. We can make a statement of the form: "There is a probability of *blah* that the true result lies between *blah* and *blah*." This is called a confidence interval.

Acknowledgements

These notes are inspired by Prof Woodhouse's notes for the same course and incorporate several suggestions from Drs Gee and Treece. I am very grateful to Martin Weber and Naveed Ahmad for checking the algebra in the examples.