# Part IA Engineering

# Mathematics

# Lent Term

# Convolution

# Fourier Series

# Probability

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# Contents & Examples Questions

# Contents & Examples Qs contd.





# Section 1

## Linear System & Impulse Response

We motivate the study of linear time-invariant systems.

The principle of superposition is explained.

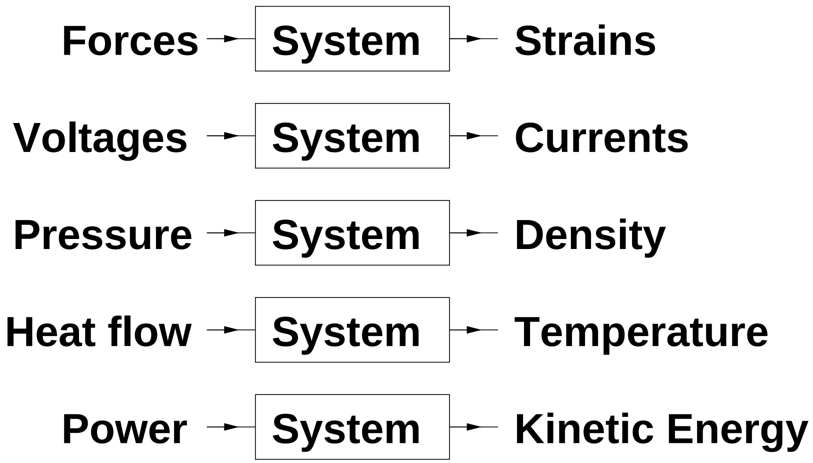
Step functions and delta functions are introduced, together with their corresponding responses.

Examples are given to illustrate the use of the step response with superposition.

The sifting theorem is stated and illustrated with some examples.

## Motivation

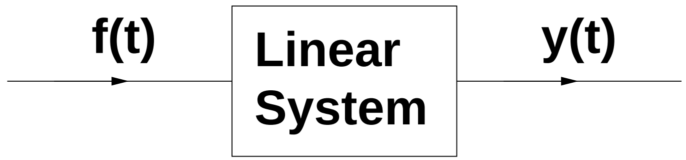
Many engineering problems concern linear systems.




In a linear system the output is computed as some linear combination of the inputs (including inputs from the past, if we are considering a system with a time-varying input and output).



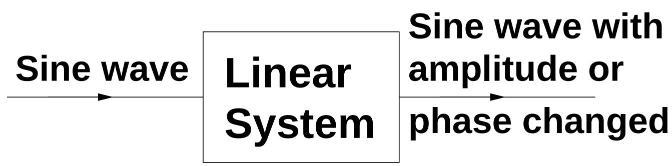
## Linear Systems



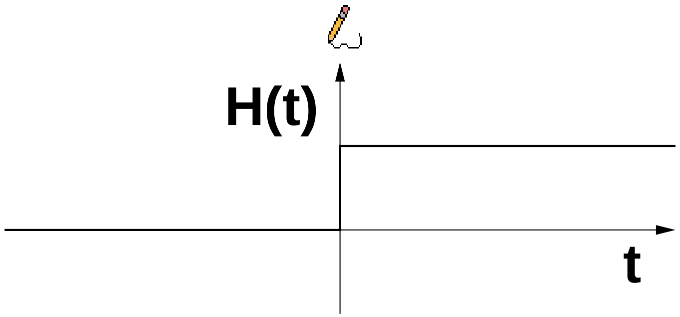

1. Linear time-invariant systems satisfy the principle of superposition.

If input $f_1(t)$ $\rightarrow$ output $y_1(t)$
and input $f_2(t)$ $\rightarrow$ output $y_2(t)$
then
input $\alpha f_1(t) + \beta f_2(t)$ $\rightarrow$ output $\alpha y_1(t) + \beta y_2(t)$
where $\alpha$ and $\beta$ are any constants.

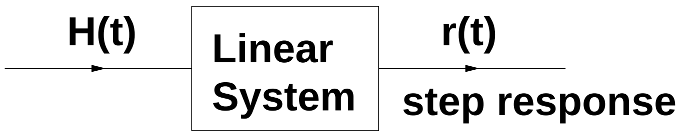
2. Sine waves have the special property that a sine wave at the input leads to a (possibly different) sine wave at the output.



# Step Function

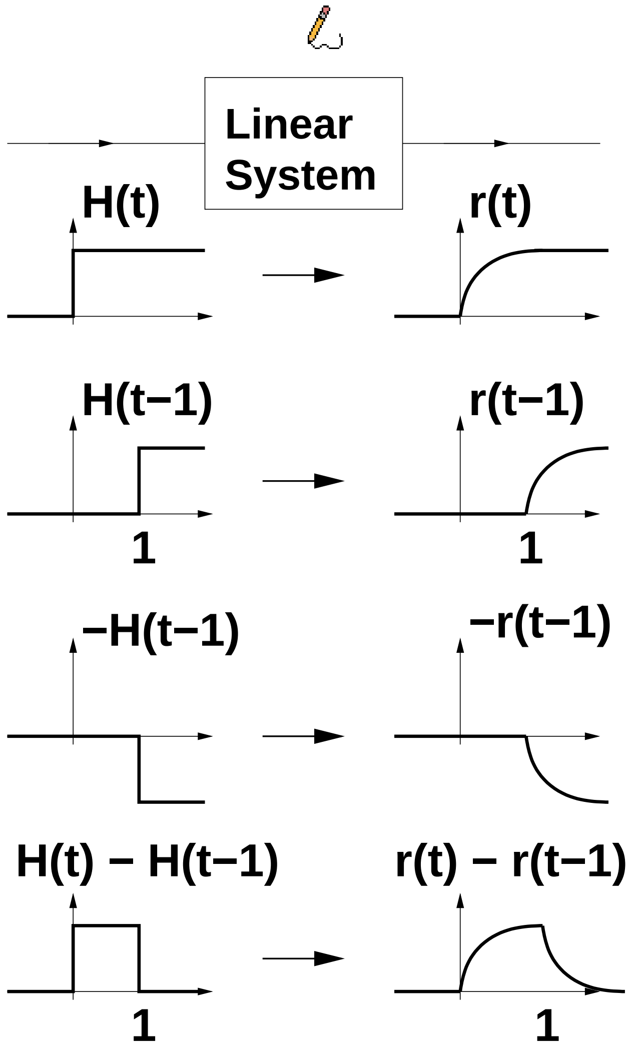



$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$






# Superposition Example

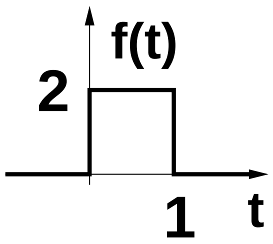


# Calculation of Superposition

Find the output of a linear system with step response

$$r(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-5t}, & t \geq 0 \end{cases}$$

when the input is the pulse $f(t)$.




**Case (a):** When $0 \leq t < 1$ the input is the same as a scaled step function so the output $y(t)$ is given by

$$y(t) = 2\left(1 - e^{-5t}\right) \qquad , 0 \leq t < 1$$

**Case (b):** When $1 \leq t$

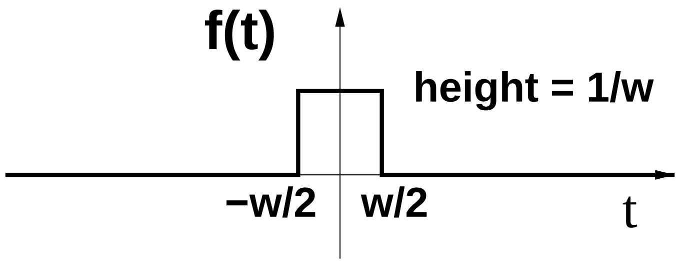
$$f(t) = 2H(t) - 2H(t-1)$$

therefore the output $y(t)$ is given by

$$\begin{aligned} y(t) &= 2r(t) - 2r(t-1) \\ &= 2\left(1 - e^{-5t} - 1 + e^{-5(t-1)}\right) \\ &= 2\left(e^{5} - 1\right)e^{-5t} \qquad , 1 \leq t \end{aligned}$$



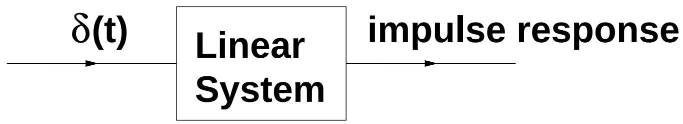
# Dirac Delta Function




As $w \to 0$ the pulse $f(t)$ becomes narrower and taller. In the limit as $w \to 0$ the pulse $f(t)$ becomes a delta function: $\delta(t)$.

The delta function is a spike with unit area. It goes bang when its argument is zero.

$$\begin{aligned} \delta(t) &= 0 \quad \text{except at } t = 0 \\ \int_a^b \delta(t)dt &= 1 \quad \text{provided } a < 0 \text{ and } b > 0 \end{aligned}$$



# Integrating the Delta Function

From the previous page

$$\int_a^b \delta(t)\,dt = 1 \quad \text{provided } a < 0 \text{ and } b > 0$$

thus

$$\begin{aligned}\int_{-\infty}^{T} \delta(t)\,dt &= \begin{cases} 0, & T < 0 \\ 1, & T > 0 \end{cases} \\ &= H(T)\end{aligned}$$

The integral of a delta function is a step function.

Conversely, the derivative of a step function is a delta function.

**Delta Function δ (t)** —integrate→ **Step Function H(t)**

**Delta Function δ (t)** ←differentiate— **Step Function H(t)**



# Impulse Response

**Impulse Response g(t)** —integrate→ **Step Response r(t)**

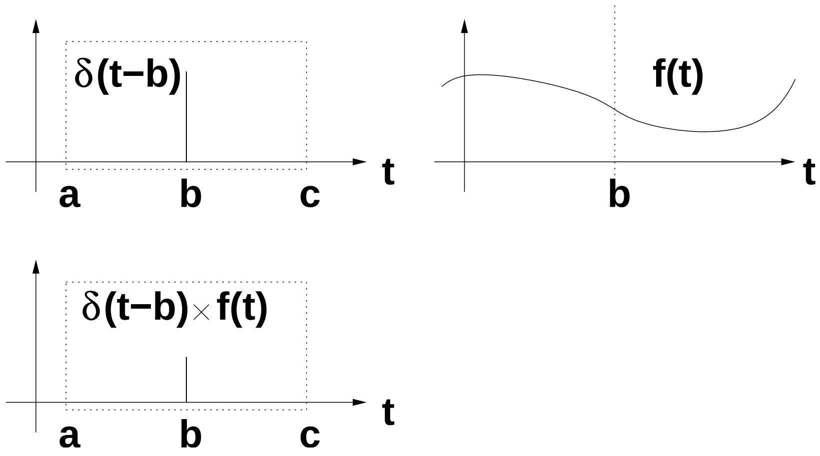
**Impulse Response g(t)** ←differentiate— **Step Response r(t)**

Find the output, $g(t)$ of a linear system with step response $r(t) = 1 - e^{-5t}$ when the input is the delta function $\delta(t)$.

$$\begin{aligned} r(t) &= 1 - e^{-5t} \\ g(t) &= \frac{dr}{dt} = 5e^{-5t} \end{aligned}$$

So the impulse response of the system is $5e^{-5t}$.

# Sifting Theorem




$$\int_a^c \delta(t-b)dt = 1 \quad \text{provided } a < b \text{ and } c > b$$

$$\int_a^c \delta(t-b)f(t)dt = f(b) \quad \text{provided } a < b \text{ and } c > b$$



# Sifting Examples

$$\int_{-\pi}^{\pi} \cos(2t)\,\delta(t)\,dt = \cos(0) = 1$$

$$\int_{-\pi}^{\pi} \cos(2t)\,\delta\left(t - \frac{\pi}{2}\right)\,dt = \cos(\pi) = -1$$

$$\int_{-\pi}^{0} \cos(2t)\,\delta\left(t - \frac{\pi}{2}\right)\,dt = 0$$

$$\int_{-\pi}^{\pi} t\,\delta\left(t + \frac{\pi}{2}\right)\,dt = -\frac{\pi}{2}$$

$$\int_{0}^{\pi} t\,\delta\left(t + \frac{\pi}{2}\right)\,dt = 0$$

# Section 1: Summary

Superposition:

If input $f_1(t)$ $\rightarrow$ output $y_1(t)$
and input $f_2(t)$ $\rightarrow$ output $y_2(t)$ then
input $\alpha f_1(t) + \beta f_2(t)$ $\rightarrow$ output $\alpha y_1(t) + \beta y_2(t)$
where $\alpha$ and $\beta$ are any constants.

Sifting:

$$\int_a^c \delta(t-b) f(t) dt = f(b) \quad \text{provided } a < b \text{ and } c > b$$

Step function and step response.

Impulse function and impulse response.

Finding the system response to a pulse by combining scaled and delayed step responses using superposition.



# Section 2

# Differential Equations to

# Describe Linear Systems

We motivate the convolution integral, which will be presented in section 3, using an example of a car going up a step.

A technique is described for solving a linear differential equation to obtain the step response of the system. We set the input to 1, and solve with initial conditions $y = \dot{y} = 0$ for $t = 0$. The impulse response can then be obtained by differentiating the step response.

The utility of this technique, when used together with convolution, is outlined.

# Differential Equations

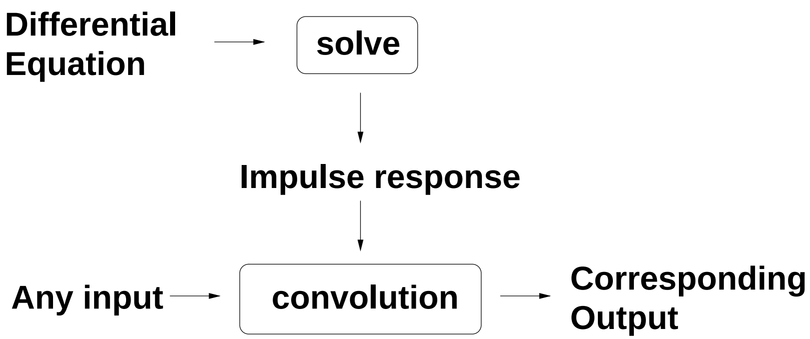
Linear systems are often described using differential equations. For example:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = f(t)$$

where f(t) is the input to the system and y(t) is the output.

We know how to solve for $y$ given a specific input $f$.

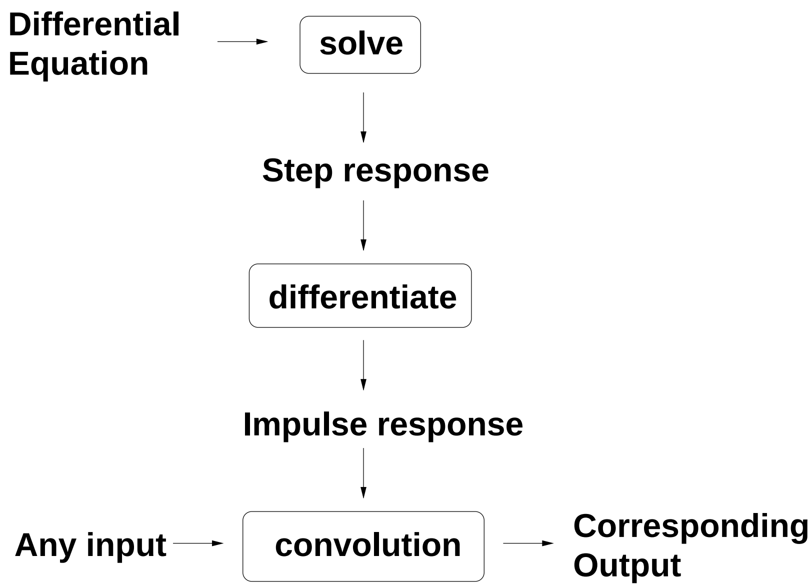
We now cover an alternative approach:





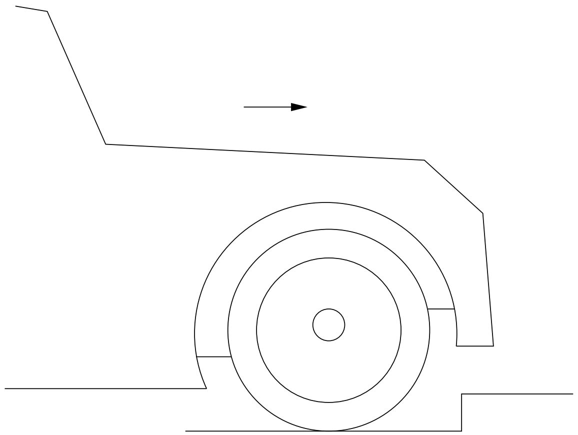

# Solving for Impulse Response

We cannot solve for the impulse response directly so we solve for the step response and then differentiate it to get the impulse response.



# Motivation: Convolution

If we know the response of a linear system to a step input, we can calculate the impulse response and hence we can find the response to any input by convolution.



Suppose we want to know how a car's suspension responds to lots of different types of road surface.

We measure how the suspension responds to a step input (or calculate the step response from a theoretical model of the system).

We can then find the impulse response and use convolution to find the car's behaviour for any road surface profile.



# Solving for Step Response

If we want to find the step response of

$$\frac{dy}{dt} + 5y = f(t)$$

where $f$ is the input and $y$ is the output. It would be nice if we could put $f(t) = H(t)$ and solve. Unfortunately we don't know of a way to do this directly. So we

1. set $f(t) = 1$, and solve for just $t \geq 0$
2. set the boundary condition $y(0) = 0$ (also $\dot{y}(0) = 0$ for second order equations) to imply that $f(t)$ was zero for all $t < 0$.

We thus have a complete solution because $y = 0$ for $t < 0$, and we have found $y$ for all $t \geq 0$.