# Differential Equations

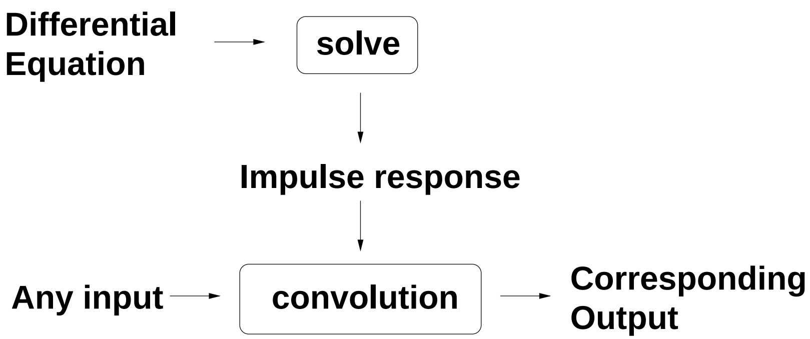
Linear systems are often described using differential equations. For example:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = f(t)$$

where f(t) is the input to the system and y(t) is the output.

We know how to solve for $y$ given a specific input $f$.

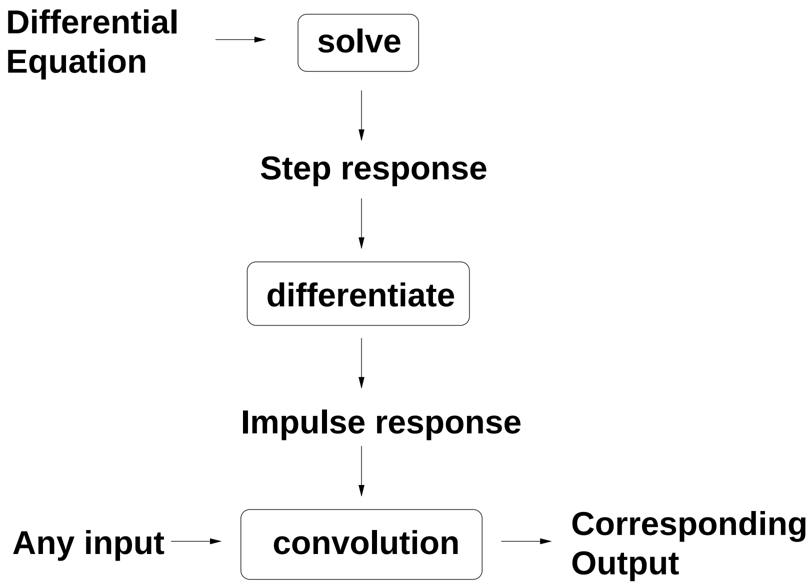
We now cover an alternative approach:





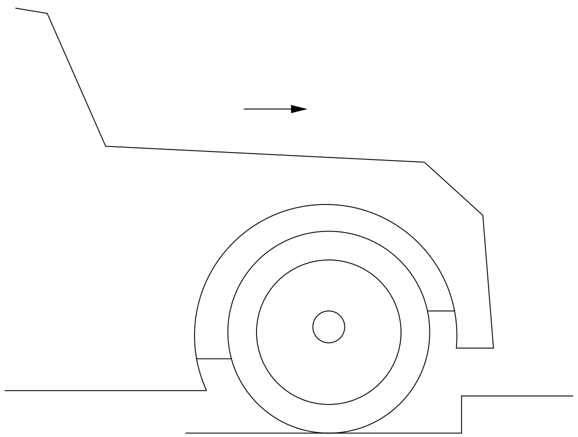

# Solving for Impulse Response

We cannot solve for the impulse response directly so we solve for the step response and then differentiate it to get the impulse response.



# Motivation: Convolution

If we know the response of a linear system to a step input, we can calculate the impulse response and hence we can find the response to any input by convolution.



Suppose we want to know how a car's suspension responds to lots of different types of road surface.

We measure how the suspension responds to a step input (or calculate the step response from a theoretical model of the system).

We can then find the impulse response and use convolution to find the car's behaviour for any road surface profile.



# Solving for Step Response

If we want to find the step response of

$$\frac{dy}{dt} + 5y = f(t)$$

where $f$ is the input and $y$ is the output. It would be nice if we could put $f(t) = H(t)$ and solve. Unfortunately we don't know of a way to do this directly. So we

1. set $f(t) = 1$, and solve for just $t \geq 0$
2. set the boundary condition $y(0) = 0$ (also $\dot{y}(0) = 0$ for second order equations) to imply that $f(t)$ was zero for all $t < 0$.

We thus have a complete solution because $y = 0$ for $t < 0$, and we have found $y$ for all $t \geq 0$.

# Boundary Condition Justification

Prove that $y = 0$ at $t = 0$ by contradiction.

We know that $y(t) = 0$ for all $t < 0$. Therefore the only way for $y$ to equal something other than zero at $t = 0$ is if there is a step discontinuity in $y$ at $t = 0$.

Assume that $y$ has a step of height $h$ at $t = 0$. If $y$ has a step discontinuity at $t = 0$ then $\frac{dy}{dt}$ must have a delta function at $t = 0$.

So we have:

- $f(t)$ is a step function so $|f(t)| \leq 1$ for all $t$.
- $|y| \leq h$ at $t = 0$.
- $\left|\frac{dy}{dt}\right| \to \infty$ at $t = 0$.

Which violates the original equation at $t = 0$.

$$\frac{dy}{dt} = f(t) - 5y$$

As the RHS is finite but the LHS is infinite. Therefore $y$ must be continuous at $t = 0$, and we can use the initial condition $y(0) = 0$.



# Step Response Example

**Step 1: set $f(t) = 1$, and solve for just $t \geq 0$.**

$$\frac{dy}{dt} + 5y = 1$$

Complimentary function: $\dot{y} + 5y = 0 \Rightarrow y = Ae^{-5t}$

Particular Integral: try $y = \lambda$ (a const) $\Rightarrow y = \frac{1}{5}$

General Solution: $y = Ae^{-5t} + \frac{1}{5}$

**Step 2: set the boundary condition $y = 0$ at $t = 0$**

$y(0) = 0 \Rightarrow A + \frac{1}{5} = 0 \Rightarrow A = -\frac{1}{5}$

So step response is $y(t) = \frac{1}{5}\left(1 - e^{-5t}\right)$ for $t \geq 0$.

# Step $\longrightarrow$ Impulse Response

**Impulse Response g(t)** —— **integrate** ⟶ **Step Response**

**Impulse Response g(t)** ⟵ **differentiate** —— **Step Response**

Step response is $y(t) = \frac{1}{5}\left(1 - e^{-5t}\right)$ for $t \geq 0$.

Impulse response $g(t)$ is given by:

$$g(t) = \begin{cases} 0,\ t < 0 \\ \frac{d}{dt}\left[\frac{1}{5}\left(1 - e^{-5t}\right)\right] = e^{-5t},\ t \geq 0 \end{cases}$$



# Find the Impulse Response

$$\frac{d^2y}{dt^2} + 13\frac{dy}{dt} + 12y = f(t)$$

**1.** Find the General Solution with $f(t) = 1$

Complimentary function is $y = Ae^{-12t} + Be^{-t}$

Particular integral is $y = \frac{1}{12}$

General solution is $y = \frac{1}{12} + Ae^{-12t} + Be^{-t}$

**2.** Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.

$$\begin{aligned} &\tfrac{1}{12} + A + B = 0 \\ &-12A - B = 0 \\ &\Rightarrow A = \tfrac{1}{132} \text{ and } B = -\tfrac{1}{11} \end{aligned}$$

Thus Step Response is $y = \frac{1}{12} + \frac{e^{-12t}}{132} - \frac{e^{-t}}{11}$

**3.** Differentiate the step response to get the impulse response.

$$g(t) = \frac{dy}{dt} = \frac{e^{-t} - e^{-12t}}{11} \qquad , (t > 0)$$

# Using the Impulse Response

If we have a system input composed of impulses,

$$f(t) = 3\delta(t-1) + 4\delta(t-2)$$

we can find the corresponding system output using superposition.

$$\begin{aligned} y(t) &= 3g(t-1) + 4g(t-2) \\ &= 3\left[\frac{e^{-(t-1)} - e^{-12(t-1)}}{11}\right] \\ &\quad +4\left[\frac{e^{-(t-2)} - e^{-12(t-2)}}{11}\right] \end{aligned}$$



# More General Input

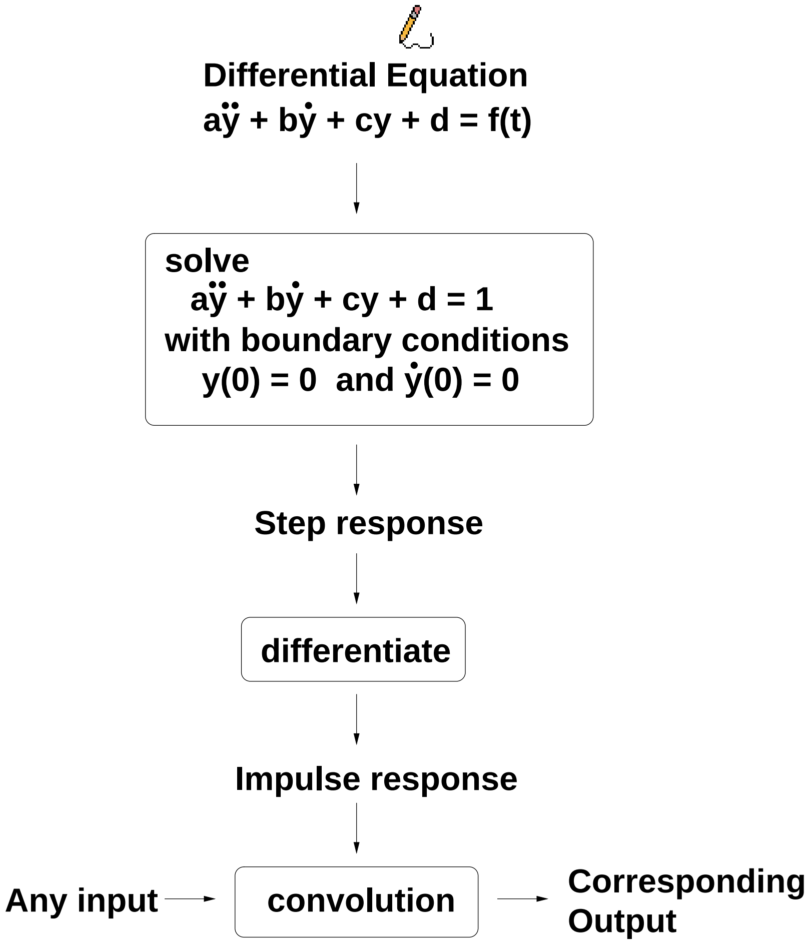
Suppose our input is composed of lots of delta functions:

$$f(t) = \sum_n p_n\,\delta(t-q_n)$$

Then the corresponding system output will be

$$y(t) = \sum_n p_n\,g(t-q_n)$$

# Section 2: Summary






# Section 3

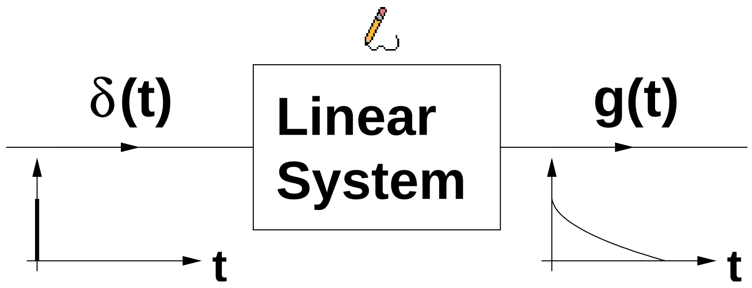
# Convolution

In this section we derive the convolution integral and show its use in some examples.

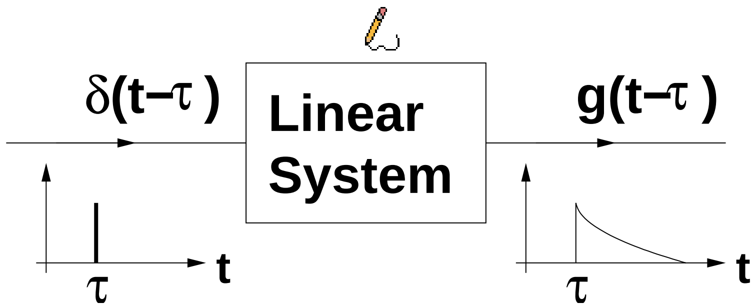
# Convolution

Our goal is to calculate the output, $y(t)$ of a linear system using the input, $f(t)$, and the impulse response of the system, $g(t)$.

An impulse at time $t = 0$ produces the impulse response.

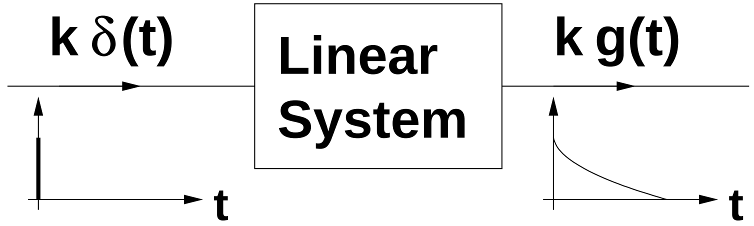



An impulse delayed to time $t = \tau$ produces a delayed impulse response starting at time $\tau$.

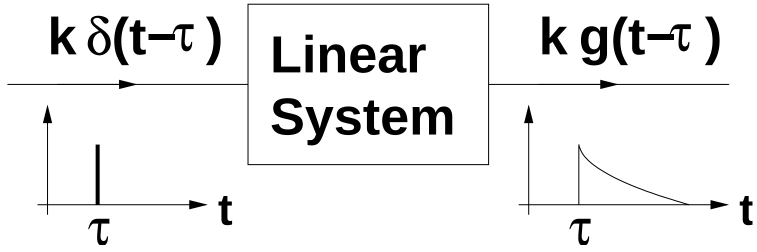




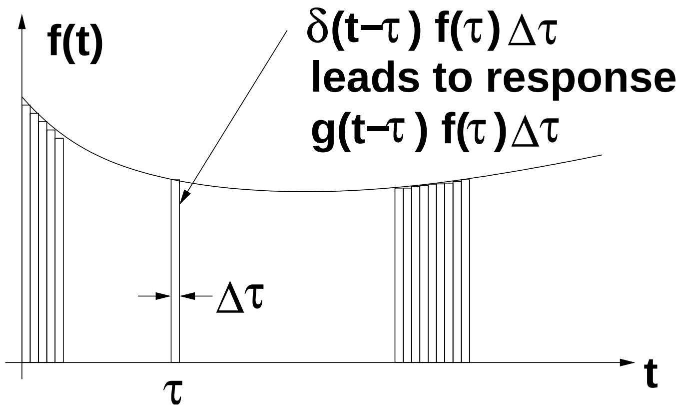

A scaled impulse at time $t = 0$ produces a scaled impulse response.




An impulse that has been scaled by $k$ and delayed to time $t = \tau$ produces an impulse response scaled by $k$ and starting at time $\tau$.



Consider the input, $f(t)$ to be made up of a sequence of strips of width $\Delta\tau$. Each of these strips is similar to a delta function and thus leads to a system output of an appropriately scaled and delayed impulse response.




The response of the system, $y(t)$ is thus the sum of these delayed, scaled impulse responses. (Provided $g(t) = 0$ for $t < 0$.)

$$y(t) \approx \sum_{\text{All slices}} g(t-\tau) f(\tau) \Delta\tau$$

Let the width of the slices tend to zero. The sum turns into an integral called the convolution integral.

$$y(t) = \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau$$



$$y(t) = \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau$$

- Treat $t$ as a constant when evaluating the integral. The integration variable is $\tau$.
- $t$ is time as it relates to the output of the system $y(t)$.
- $\tau$ is time as it relates to the input of the system $f(\tau)$.

# Convolution Example 1

Consider a system with impulse response

$$g(t) = \begin{cases} 0 & , t < 0 \\ e^{-5t} & , t \geq 0 \end{cases}$$

Find the output for input $f(t) = H(t)$ (step function).

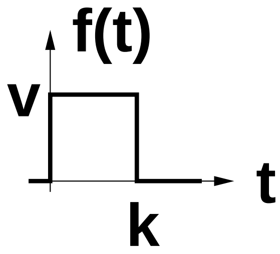
$$\begin{aligned} y(t) &= \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau \\ &= \int_{-\infty}^{t} e^{-5(t-\tau)} H(\tau) d\tau \\ &= \int_{0}^{t} e^{-5(t-\tau)} d\tau \\ &= \left[\frac{1}{5} e^{-5(t-\tau)}\right]_0^t \\ &= \frac{1}{5}\left(1 - e^{-5t}\right) \end{aligned}$$



# Convolution Example 2

For the same system ($g(t) = e^{-5t}$, $t \geq 0$), find the output for input

$$f(t) = \begin{cases} 0, & t < 0 \\ v, & 0 < t < k \\ 0, & t > k \end{cases}$$




Using the convolution integral, the answer is given by

$$y(t) = \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau$$

$$= \begin{cases} \int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau, & t < 0 \\ \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\ \quad + \int_{0}^{t} g(t-\tau)\, v \, d\tau, & 0 < t < k \\ \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\ \quad + \int_{0}^{k} g(t-\tau)\, v \, d\tau \\ \quad + \int_{k}^{t} g(t-\tau) \times 0 \, d\tau, & t > k \end{cases}$$

**Case (a):** $t < 0$

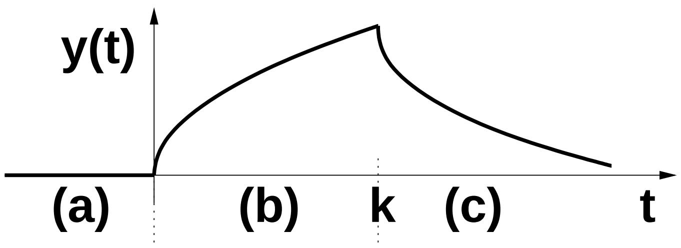
$\int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau = 0$ so $y(t) = 0$ for all $t < 0$.

**Case (b):** $0 < t < k$

$$
\begin{aligned}
y(t) = \int_0^t g(t-\tau)\, v \, d\tau &= \int_0^t e^{-5(t-\tau)}\, v \, d\tau \\
&= \frac{v}{5}\left[e^{-5(t-\tau)}\right]_0^t \\
&= \frac{v}{5}\left(1 - e^{-5t}\right)
\end{aligned}
$$

**Case (c):** $t > k$

$$
\begin{aligned}
y(t) = \int_0^k g(t-\tau)\, v \, d\tau &= \int_0^k e^{-5(t-\tau)}\, v \, d\tau \\
&= \frac{v}{5}\left[e^{-5(t-\tau)}\right]_0^k \\
&= \frac{v}{5}\left(e^{5k} - 1\right) e^{-5t}
\end{aligned}
$$

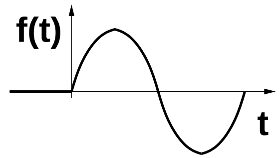





# Convolution Example 3

For the same system ($g(t) = e^{-5t}$, $t \geq 0$), find the output for input

$$
f(t) = \begin{cases} 0, & t < 0 \\ \sin(\omega t), & t > 0 \end{cases}
$$




Using the convolution integral, the answer is given by

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau \\
&= \begin{cases} \int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau, & t < 0 \\ \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\ \quad + \int_0^t g(t-\tau) \sin(\omega\tau) \, d\tau, & 0 < t \end{cases}
\end{aligned}
$$

**Case (a):** $t < 0$

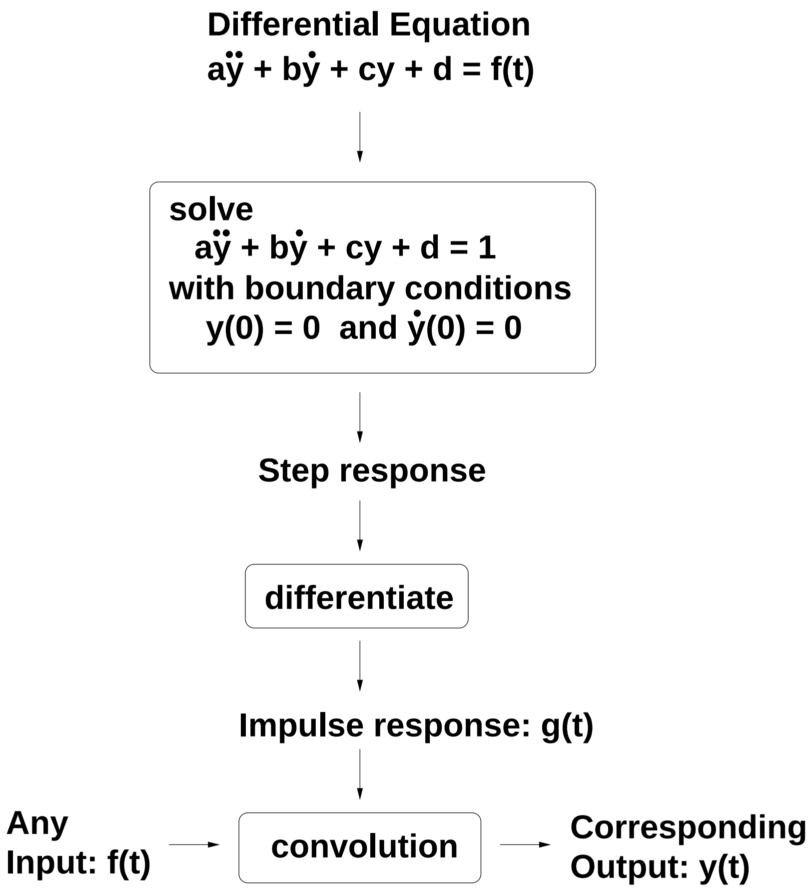
$\int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau = 0$ so $y(t) = 0$ for all $t < 0$.

**Case (b):** $0 < t$

$$
\begin{aligned}
y(t) &= \int_0^t g(t-\tau)\sin(\omega\tau)\,d\tau \\
&= \int_0^t e^{-5(t-\tau)}\sin(\omega\tau)\,d\tau \\
&= \text{Im}\left\{\int_0^t e^{-5(t-\tau)}e^{i\omega\tau}\,d\tau\right\} \\
&= \text{Im}\left\{e^{-5t}\left[\frac{e^{(5+i\omega)\tau}}{5+i\omega}\right]_0^t\right\} \\
&= \text{Im}\left\{\frac{e^{i\omega t}-e^{-5t}}{5+i\omega}\right\} \\
&= \frac{5\sin(\omega t)-\omega\cos(\omega t)+\omega e^{-5t}}{25+\omega^2}
\end{aligned}
$$



# Convolution Summary




$$y(t) = \int_{-\infty}^{t} g(t-\tau)\,f(\tau)\,d\tau$$

# Complete Example

Find the impulse response of

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = f(t)$$

hence find the output when the input $f(t) = H(t)e^{-t}$.

**1.** Find the General Solution with $f(t) = 1$

Complimentary function is $y = Ae^{-t} + Be^{-2t}$

Particular integral is $y = \frac{1}{2}$

General solution is $y = \frac{1}{2} + Ae^{-t} + Be^{-2t}$

**2.** Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.

$$\begin{aligned} &\tfrac{1}{2} + A + B = 0 \\ &-A - 2B = 0 \\ &\Rightarrow A = -1 \text{ and } B = \tfrac{1}{2} \end{aligned}$$

Thus Step Response is $y = \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}$



**3.** Differentiate the step response to get the impulse response.

$$g(t) = \frac{dy}{dt} = e^{-t} - e^{-2t}$$

**4.** Use the convolution integral to find the output for the required input.

The required input is $f(t) = e^{-t}$, $t > 0$.

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} g(t-\tau)f(\tau)d\tau \\ &= \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) e^{-\tau}\, d\tau \\ &= \int_0^t e^{-t} - e^{\tau-2t}\, d\tau \\ &= \left[\tau e^{-t} - e^{\tau-2t}\right]_0^t \\ &= (t-1)\, e^{-t} + e^{-2t} \end{aligned}$$

# Section 3: Summary

Convolution integral (memorise this):

$$
\begin{aligned}
f(t) &= \text{input} \\
g(t) &= \text{impulse response} \\
y(t) &= \text{output} \\
y(t) &= \int_{-\infty}^{t} g(t-\tau)\, f(\tau)\, d\tau
\end{aligned}
$$

Way to find the output of a linear system, described by a differential equation, for an arbitrary input:

- Find general solution to equation for input = 1.
- Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.
- Differentiate to get the impulse response.
- Use convolution integral together with the impulse response to find the output for any desired input.



# Section 4

# Evaluating Convolution Integrals

A way of rearranging the convolution integral is described and illustrated.

The differences between convolution in time and space are discussed and the concept of causality is introduced.

The section ends with an example of spatial convolution.