Section 5

Fourier Series

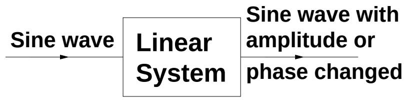
The Fourier series is introduced using an analogy with splitting vectors up into components.

The symmetry properties that enable us to predict that certain coefficients are zero are presented.

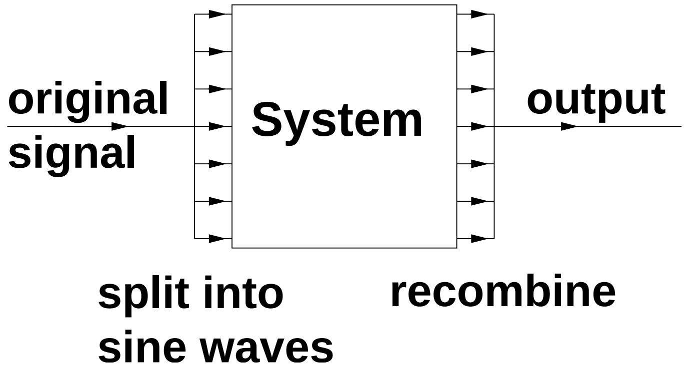
Motivation

We mentioned at the start of the last section that sine waves have a special property in relation to linear systems.

A sine wave at the input leads to a (possibly different) sine wave at the output.

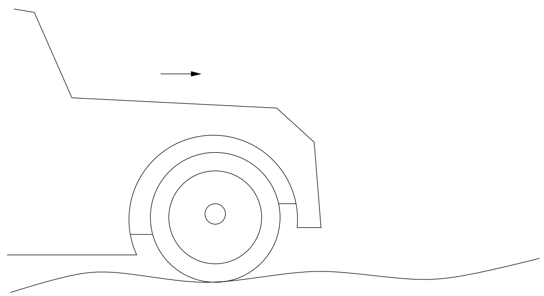


It would therefore be useful to be able to express an arbitrary signal in terms of a sum of sine waves.



Motivation: Car Suspension

Supposing we know that our car suspension will start to oscillate (bounce up and down uncomfortably) at frequency f.



We want to measure a variety of typical road profiles and calculate how much of frequency f they each contain (with the car travelling at a particular speed).

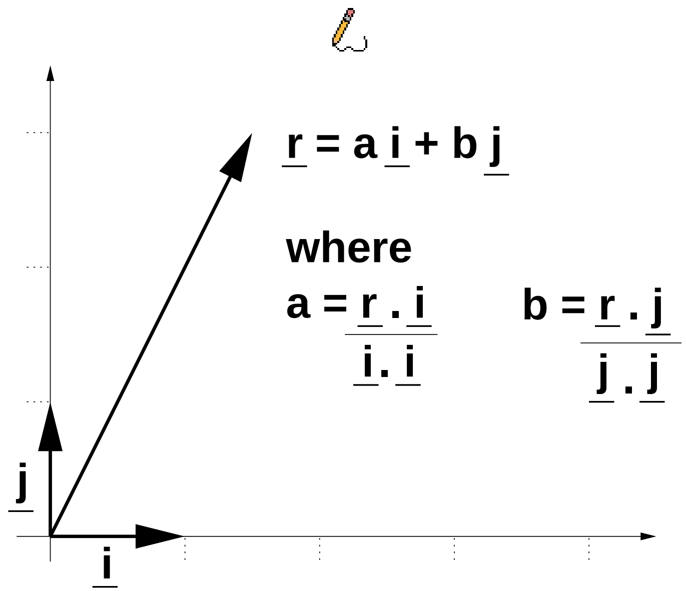
This will tell us which combinations of road profile and speed are likely to be a problem.

The Fourier series enables us to represent the road profile as the sum of a set of sinusoidal components at different frequencies.

Splitting up Vectors

We want to express a signal $f(t)$ in the range $-\pi \leq t \leq \pi$ in terms of some basic signals, i.e. sine waves. Let's look first at how we do a similar thing with vectors.

Consider how we express the arbitrary vector $\underline{r}$ in terms of the basis vectors $\underline{i}$ and $\underline{j}$.



The basis vectors are orthogonal: $\underline{i}.\underline{j} = 0$.

Basis Functions

Just as we represent $\underline{r}$ using orthogonal basis vectors, we want to represent $f(t)$ in the range $-\pi$ to π using orthogonal basis functions. We only need two vectors, but we need an infinite number of functions.

1 (i.e. a constant term)
$\cos(t) \quad \cos(2t) \quad \cos(3t) \quad \cos(4t) \quad \ldots$
$\sin(t) \quad \sin(2t) \quad \sin(3t) \quad \sin(4t) \quad \ldots$

If n and m are positive integers greater than zero.

$$\int_{-\pi}^{\pi} \cos(nt)\sin(mt)\,dt = 0$$

$$\int_{-\pi}^{\pi} \cos(nt) \times 1\,dt = 0$$

$$\int_{-\pi}^{\pi} \sin(nt) \times 1\,dt = 0$$

$$\int_{-\pi}^{\pi} \cos(nt)\cos(mt)\,dt = \begin{cases} 0 & , n \neq m \\ \pi & , n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nt)\sin(mt)\,dt = \begin{cases} 0 & , n \neq m \\ \pi & , n = m \end{cases}$$

$$\int_{-\pi}^{\pi} 1 \times 1\,dt = 2\pi$$

So, using $\int_{-\pi}^{\pi} p(t)q(t)\,dt$ as our "dot product for functions", the basis functions are orthogonal.

Fourier Series

The equivalents of our vector dot product expressions to calculate the component of $\underline{r}$ in each direction (eg. $a = (\underline{r}.\underline{i})/(\underline{i}.\underline{i})$) are:

$$a_n = \frac{\int_{-\pi}^{\pi} \cos(nt) f(t)\,dt}{\int_{-\pi}^{\pi} \cos(nt)\cos(nt)\,dt} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt) f(t)\,dt$$

$$b_n = \frac{\int_{-\pi}^{\pi} \sin(nt) f(t)\,dt}{\int_{-\pi}^{\pi} \sin(nt)\sin(nt)\,dt} = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt) f(t)\,dt$$

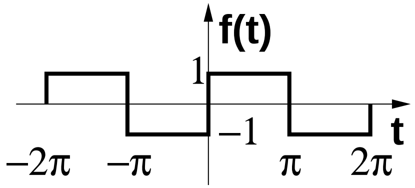
$$d = \frac{\int_{-\pi}^{\pi} 1 \times f(t)\,dt}{\int_{-\pi}^{\pi} 1 \times 1\,dt} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

The equivalent of our vector expression for $\underline{r}$ in terms of $\underline{i}$ and $\underline{j}$, (i.e. $\underline{r} = a\underline{i} + b\underline{j}$) is an expression for f in terms of all the basis functions.

$$f(t) = \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) + d \times 1$$

Fourier Series Example 1

Represent the square wave $f(t)$ as a Fourier series.



$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt) f(t)\, dt &= 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt) f(t)\, dt &= \frac{2\,(1-(-1)^n)}{n\pi} \\
d &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt &= 0
\end{aligned}
$$

Thus, we can model the square wave function $f(t)$ using:

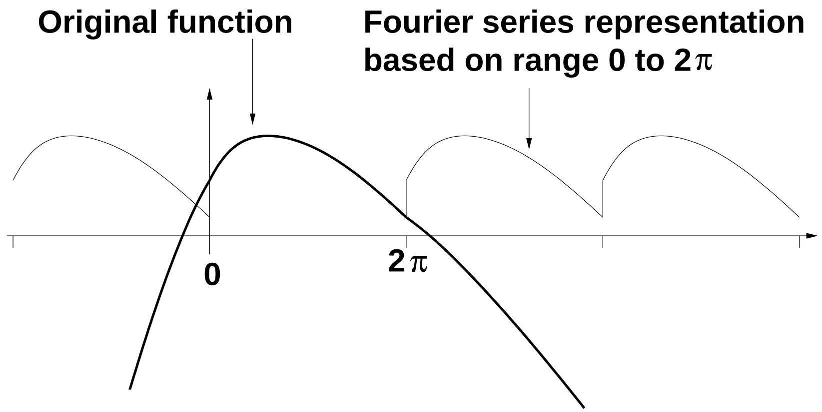
$$
\begin{aligned}
f(t) &= d + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right) \\
&= \sum_{n=1}^{\infty} \frac{2\,(1-(-1)^n)}{n\pi} \sin(nt) \\
&= \frac{4}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]
\end{aligned}
$$

Fourier Series Properties

1. We can use any range of length 2π instead of $-\pi \le t \le \pi$ in the Fourier formulae. For example, $0 \le t \le 2\pi$ is equally OK.

2. We are only modelling the function $f(t)$ in the specified range (eg. $-\pi$ to π, or 0 to 2π). Outside this range the model will just repeat with period 2π.

This is fine if the function we wish to model is periodic itself, but if the function is not periodic the Fourier model will probably only be useful over the range on which it was built.



Fourier Series Example 2

Represent $f(t) = e^t$ as a Fourier series between $-\pi$ and π.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt)e^t\,dt = \frac{(-1)^n\left(e^{\pi}-e^{-\pi}\right)}{\pi\left(1+n^2\right)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt)e^t\,dt = \frac{-(-1)^n\left(e^{\pi}-e^{-\pi}\right)n}{\pi\left(1+n^2\right)}$$

$$d = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^t\,dt = \frac{e^{\pi}-e^{-\pi}}{2\pi}$$

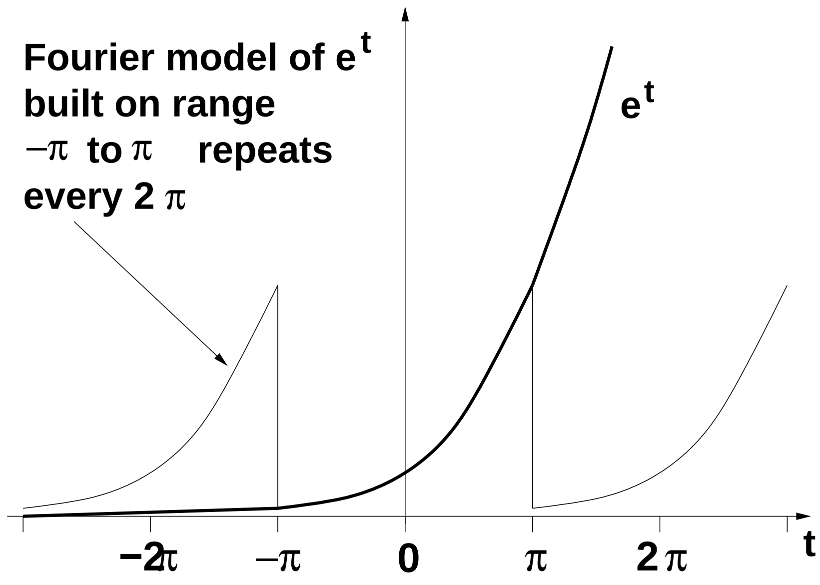
Thus, in the range $-\pi < t < \pi$ we can model the function $f(t) = e^t$ using:

$$f(t) = d + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

$$= \frac{e^{\pi}-e^{-\pi}}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\left[\cos(nt) - n\sin(nt)\right]\right)$$

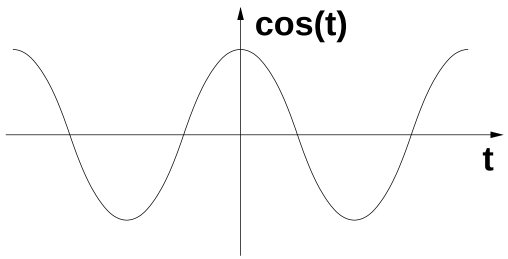
$$\approx 3.68 - 3.68\cos(t) + 3.68\sin(t) + 1.47\cos(2t) - 2.94\sin(2t) - \ldots$$

Fourier Model of Exponential

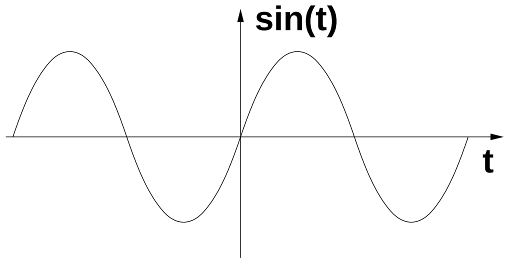


Symmetric Signals

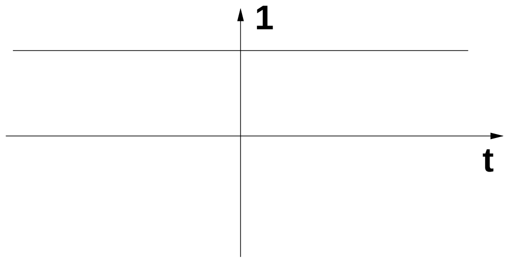
ODD function	$f(-t) = -f(t)$	eg: $\sin(t)$
EVEN function	$f(-t) = f(t)$	eg: $\cos(t)$



The a_n terms model the EVEN component in the function



The b_n terms model the ODD component in the function

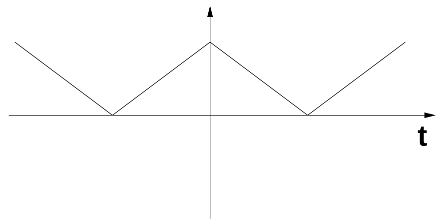


The d term models the mean value of the function

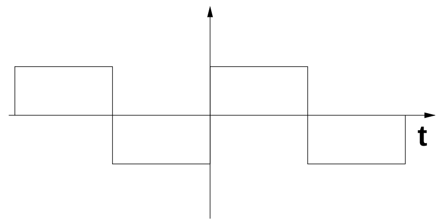
Avoiding Integration

If we can spot a symmetry in the function to be represented then we can avoid evaluating one or more of the Fourier integrals.

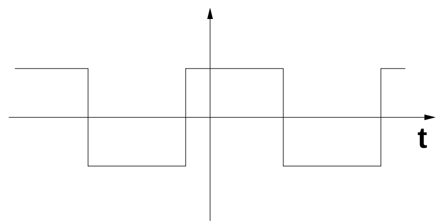
No even component $\Rightarrow$ all $a_n = 0$
No odd component $\Rightarrow$ all $b_n = 0$
Zero mean $\Rightarrow$ $d = 0$



EVEN function with non−zero mean: $b_n = 0$



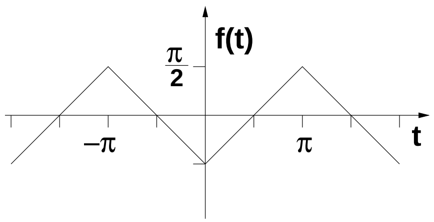
Purely ODD function with zero mean: $a_n = 0$ and $d = 0$



Function with zero mean: $d = 0$

Fourier Series Example 3

Find the Fourier series representation for the function $f(t)$ below.



EVEN function with zero mean: $b_n = 0$ and $d = 0$

We only have to calculate a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt) f(t)\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \cos(nt)(-t - \pi/2)\, dt + \frac{1}{\pi}\int_{0}^{\pi} \cos(nt)(t - \pi/2)\, dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \cos(nt)(t - \pi/2)\, dt$$

$$= \frac{2}{n^2\pi}((-1)^n - 1) = \begin{cases} 0 & , n \text{ even} \\ \frac{-4}{n^2\pi} & , n \text{ odd} \end{cases}$$

so the Fourier series is:

$$f(t) = \frac{-4}{\pi}\left[\cos(t) + \frac{1}{9}\cos(3t) + \frac{1}{25}\cos(5t) + \dots\right]$$

Fourier Series Example 4

Find the Fourier series representation for the function $f(t) = \cos(t + \pi/4)$.

This function has a mean value of zero so $d = 0$.

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt)\cos(t + \pi/4)\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nt + t + \frac{\pi}{4}) + \cos(nt - t - \frac{\pi}{4})\, dt \\ &= \frac{1}{\sqrt{2}}, \text{ when } n = 1 \text{ and 0 otherwise.} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt)\cos(t + \pi/4)\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nt + t + \frac{\pi}{4}) + \sin(nt - t - \frac{\pi}{4})\, dt \\ &= \frac{-1}{\sqrt{2}}, \text{ when } n = 1 \text{ and 0 otherwise.} \end{aligned}$$

so the Fourier series is:

$$f(t) = \frac{\cos(t) - \sin(t)}{\sqrt{2}}$$

Section 5: Summary

Periodic functions, (so far only with period 2π), can be represented using the the Fourier series.

We can use symmetry properties of the function to spot that certain Fourier coefficients will be zero, and hence avoid performing the integral to evaluate them.

- Functions with zero mean have $d = 0$.
- Purely odd functions have $a_n = 0$.
- Purely even functions have $b_n = 0$.

Segments of non-periodic functions can be represented using the Fourier series in the same way. The Fourier series representation just repeats outside the range on which it was built.

Section 6

General Fourier Series

The Fourier series for arbitrary period is presented.

We compare three techniques for calculating a general range Fourier series: direct integration, using a related series of delta functions, and using the electrical data book.

During the direct integration example, some symmetry arguments for simplifying integrals are illustrated.