Section 5: Summary

Periodic functions, (so far only with period 2π), can be represented using the the Fourier series.

We can use symmetry properties of the function to spot that certain Fourier coefficients will be zero, and hence avoid performing the integral to evaluate them.

- Functions with zero mean have $d = 0$.
- Purely odd functions have $a_n = 0$.
- Purely even functions have $b_n = 0$.

Segments of non-periodic functions can be represented using the Fourier series in the same way. The Fourier series representation just repeats outside the range on which it was built.

Section 6

General Fourier Series

The Fourier series for arbitrary period is presented.

We compare three techniques for calculating a general range Fourier series: direct integration, using a related series of delta functions, and using the electrical data book.

During the direct integration example, some symmetry arguments for simplifying integrals are illustrated.

General Range

If we want to model a periodic signal with period other than 2π, or a section of a non-periodic signal of length other than 2π we need a more general formula.

To model a function $f(x)$ over the range 0 to L, substitute $\frac{2\pi x}{L} = t$, $\left(\Rightarrow \frac{2\pi}{L}dx = dt\right)$ in our Fourier formulae.

$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{2\pi n x}{L}\right) f(x)\,dx$$

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\,dx$$

$$d = \frac{1}{L}\int_0^L f(x)\,dx$$

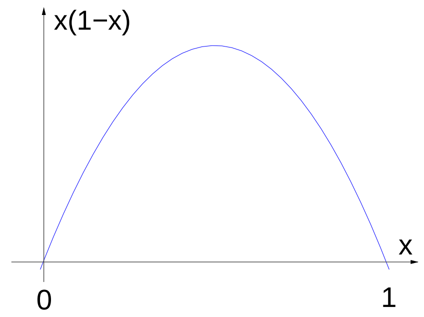
$$f(x) = d + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right]$$

The fraction $\frac{2\pi}{L}$ is often written as ω_0 and called the

fundamental angular frequency.

General Range Example 1

Represent the signal $f(x) = x(1-x)$ as a Fourier series with period 1, based on the range 0 to 1.



$$a_n = 2\int_0^1 \cos(2\pi n x)x(1-x)\,dx = \frac{-1}{n^2\pi^2}$$

$$b_n = 2\int_0^1 \sin(2\pi n x)x(1-x)\,dx = 0$$

$$d = \int_0^1 x(1-x)\,dx = \frac{1}{6}$$

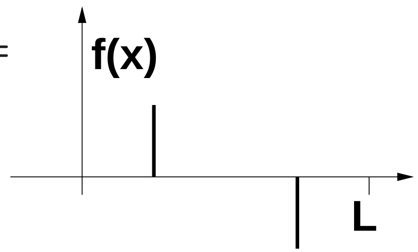
So the Fourier series is:

$$f(x) = \frac{1}{6} - \frac{\cos(2\pi x)}{\pi^2} - \frac{\cos(4\pi x)}{4\pi^2} - \frac{\cos(6\pi x)}{9\pi^2} - \ldots$$

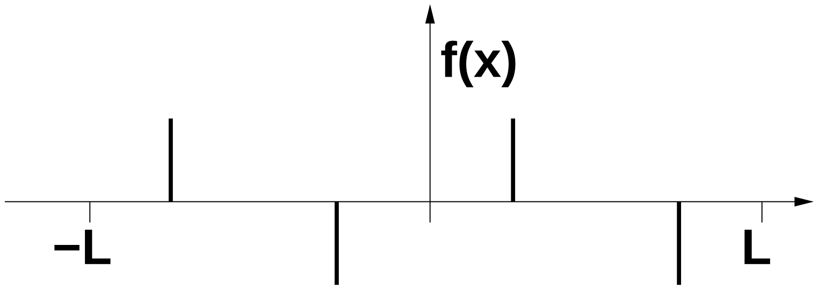
Note that this is an even function with period $= 1$.

General Range Example 2

Represent the signal $f(x) = \delta(x - L/4) - \delta(x - 3L/4)$ as a Fourier series based on the range 0 to L.



We are told that the period is L, so consider the signal repeating with period L.



This signal is purely ODD with zero mean. We therefore only need to calculate b_n.

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\,dx \\
&= \tfrac{2}{L}\int_0^L \sin\left(\tfrac{2\pi n x}{L}\right)\left[\delta\left(x - \tfrac{L}{4}\right) - \delta\left(x - \tfrac{3L}{4}\right)\right]\,dx \\
&= \frac{2}{L}\left[\sin\left(\frac{2\pi n L}{4L}\right) - \sin\left(\frac{6\pi n L}{4L}\right)\right] \qquad \text{(sifting!)} \\
&= \frac{2}{L}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right] \\
&= \frac{4}{L}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

$$b_n = \frac{4}{L} \sin\left(\frac{n\pi}{2}\right)$$

This is zero when n is even. Tabulate $\sin\left(\frac{n\pi}{2}\right)$ when n is odd.

$\sin\left(\frac{n\pi}{2}\right)$	n	$\frac{n+1}{2}$	$-1^{\left(\frac{n+1}{2}\right)}$	$\frac{n+3}{2}$	$-1^{\left(\frac{n+3}{2}\right)}$
1	1	1	−1	2	1
−1	3	2	1	3	−1
1	5	3	−1	4	1
−1	7	4	1	5	−1

Thus

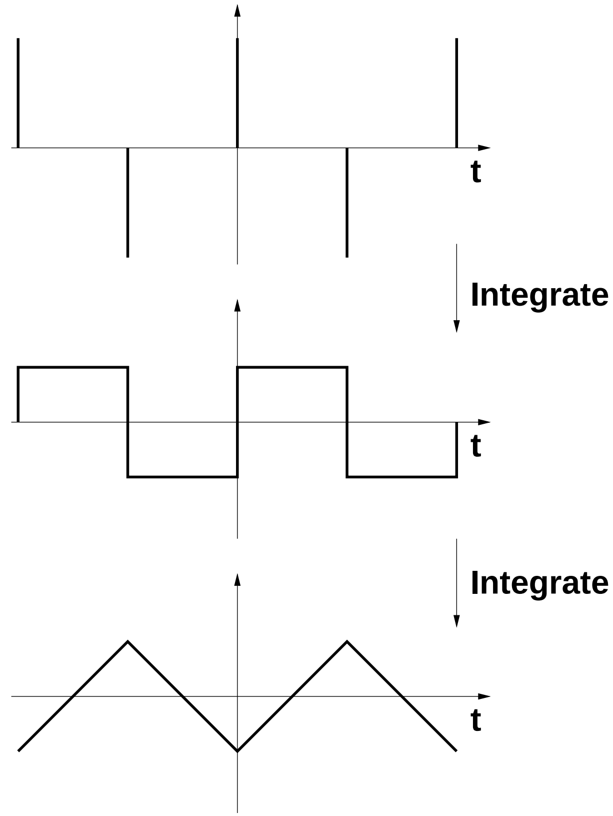
$$b_n = \begin{cases} 0 & , n \text{ even} \\ \frac{4}{L}(-1)^{\left(\frac{n+3}{2}\right)} & , n \text{ odd} \end{cases}$$

So the Fourier series is:

$$f(x) = \frac{4}{L}\left[\sin\left(\frac{2\pi x}{L}\right) - \sin\left(\frac{6\pi x}{L}\right) + \sin\left(\frac{10\pi x}{L}\right) - \ldots\right]$$

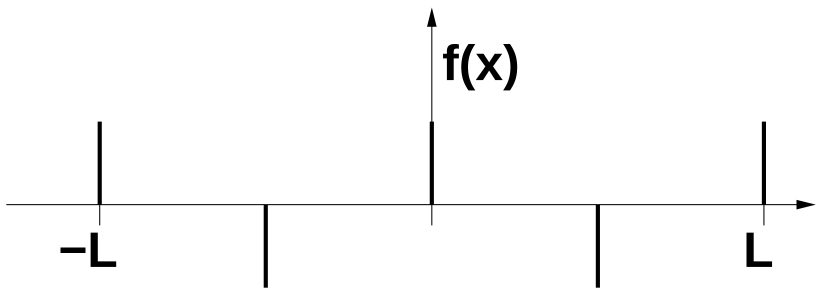
More Integral Avoidance

Notice how easy it is to calculate the Fourier series of a signal formed only of delta functions. By integrating the delta function series we can derive the Fourier series for square waves and triangle waves.



Pick the Start of Period Carefully

If you wish to find the Fourier series of a waveform such as



it is difficult to use formulae with limits such as

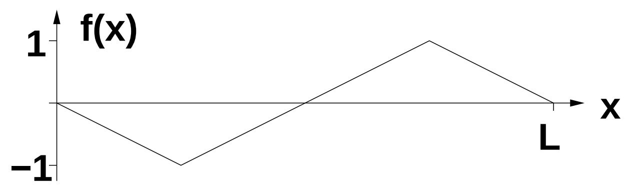
$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{2\pi n x}{L}\right) f(x)\, dx$$

because it is not clear what to do about the delta functions at that coincide with the upper and lower limits of the integral.

Instead, choose your period of length L to start at a different point. For example:

$$a_n = \frac{2}{L}\int_{\frac{-L}{4}}^{\frac{3L}{4}} \cos\left(\frac{2\pi n x}{L}\right) f(x)\, dx$$

Three Methods



There are three ways to find the Fourier series for $f(x)$ between 0 and L.

1. Use the general range Fourier formulae directly.

2. Differentiate the waveform twice to get a sequence of delta functions. Find a Fourier series for the delta functions, then integrate the series twice to get the Fourier series of the triangular wave.

3. Look up the Fourier series of a similar waveform in the Maths Data book and use a substitution of variables to find the series for the waveform we require.

Method 1: Direct Integration

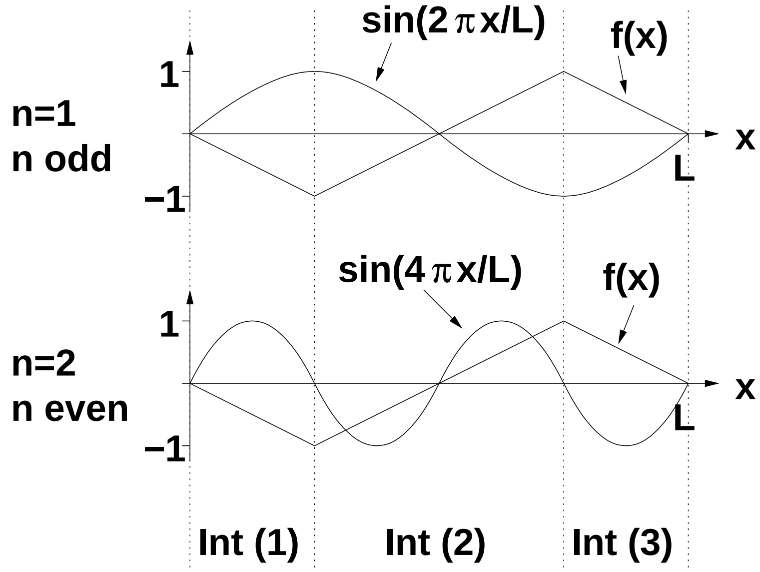
The triangular waveform is entirely ODD and has zero mean. Thus $d = 0$ and $a_n = 0$. We only need to find b_n.

To do this we need an algebraic representation of the waveform.

$$f(x) = \begin{cases} -\frac{4x}{L} & , 0 < x < \frac{L}{4} \\ \frac{4x}{L} - 2 & , \frac{L}{4} < x < \frac{3L}{4} \\ 4 - \frac{4x}{L} & , \frac{3L}{4} < x < L \end{cases}$$

From this we can write down an expression for b_n.

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\, dx \\ &= \frac{2}{L}\int_0^{\frac{L}{4}} \sin\left(\frac{2\pi n x}{L}\right)\left(\frac{-4x}{L}\right)\, dx \qquad (1) \\ &\quad +\frac{2}{L}\int_{\frac{L}{4}}^{\frac{3L}{4}} \sin\left(\frac{2\pi n x}{L}\right)\left(\frac{4x}{L} - 2\right)\, dx \qquad (2) \\ &\quad +\frac{2}{L}\int_{\frac{3L}{4}}^{L} \sin\left(\frac{2\pi n x}{L}\right)\left(4 - \frac{4x}{L}\right)\, dx \qquad (3) \end{aligned}$$



There is clearly a symmetry between the terms $f(x)$ and $\sin\left(\frac{2\pi n x}{L}\right)$.

All terms with even n are zero, and all terms with odd n are equal to twice integral (2).

When n is even $b_n = 0$ and when n is odd

$$b_n = \frac{8}{n^2\pi^2}\left(\frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right)$$

But as we know n is odd, the $\cos()$ term is always zero and we can write $\left[-\sin\left(\frac{n\pi}{2}\right)\right] = (-1)^{\left(\frac{n+1}{2}\right)}$

$$\Rightarrow b_n = \begin{cases} 0, & n \text{ even} \\ \frac{8}{n^2\pi^2} \times (-1)^{\left(\frac{n+1}{2}\right)}, & n \text{ odd} \end{cases}$$

Giving a final Fourier series for $f(x) =$

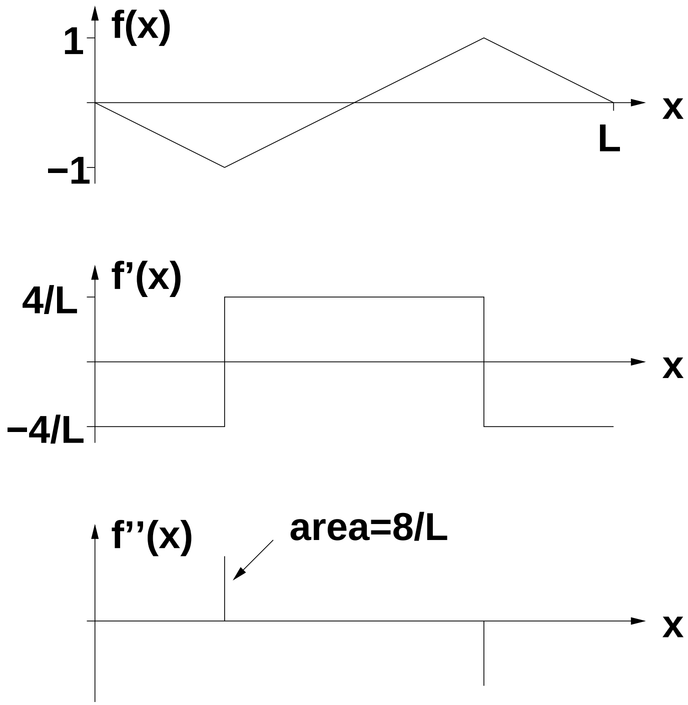
$$\frac{8}{\pi^2}\left[-\sin\left(\frac{2\pi x}{L}\right) + \frac{\sin\left(\frac{2\pi 3x}{L}\right)}{9} - \frac{\sin\left(\frac{2\pi 5x}{L}\right)}{25} + \ldots\right]$$

If we want to write this algebraically, we need to limit n to only odd values. Let $n = 2m - 1$ with m taking integer values from 1 to ∞.

$$f(x) = \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)^2}\sin\left(\frac{2\pi x(2m-1)}{L}\right)$$

Method 2: Delta Functions

First we differentiate the waveform twice.



$f''(x)$ is a purely odd function with zero mean so we only need to calculate b_n.

$$f''(x) = \frac{8}{L}\delta\left(x - \frac{L}{4}\right) - \frac{8}{L}\delta\left(x - \frac{3L}{4}\right)$$

To find the Fourier series for $f''(x)$:

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\,dx \\ &= \tfrac{16}{L^2}\int_0^L \sin\left(\tfrac{2\pi n x}{L}\right)\left[\delta\left(x-\tfrac{L}{4}\right)-\delta\left(x-\tfrac{3L}{4}\right)\right]\,dx \\ &= \frac{16}{L^2}\left[\sin\left(\frac{2\pi n L}{4L}\right)-\sin\left(\frac{6\pi n L}{4L}\right)\right] \qquad \text{(sifting!)} \\ &= \begin{cases} 0 & , n \text{ even} \\ \frac{32}{L^2}(-1)^{\left(\frac{n+3}{2}\right)} & , n \text{ odd} \end{cases} \end{aligned}$$

So the Fourier series for $f''(x) =$

$$\frac{32}{L^2}\left[\sin\left(\frac{2\pi x}{L}\right)-\sin\left(\frac{6\pi x}{L}\right)+\sin\left(\frac{10\pi x}{L}\right)-\ldots\right]$$

We can also write this (note that $2m-1=n$).

$$f''(x)=\frac{32}{L^2}\sum_{m=1}^{\infty}(-1)^{m+1}\sin\left(\frac{2\pi x(2m-1)}{L}\right)$$

Now we integrate twice, each time setting the constant of integration to zero so we get a waveform with zero mean in each case.

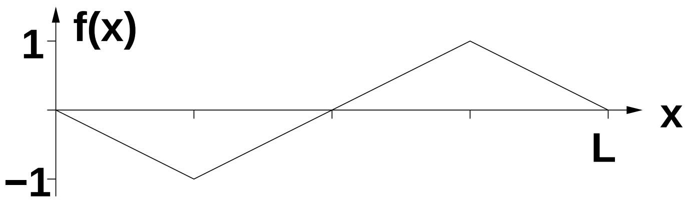
$$\begin{aligned} f''(x) &= \frac{32}{L^2}\sum_{m=1}^{\infty}\sin\left(\frac{2\pi x(2m-1)}{L}\right)(-1)^{m+1} \\ f'(x) &= \frac{16}{\pi L}\sum_{m=1}^{\infty}\frac{\cos\left(\frac{2\pi x(2m-1)}{L}\right)}{2m-1}(-1)^m \\ f(x) &= \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{2\pi x(2m-1)}{L}\right)}{(2m-1)^2}(-1)^m \end{aligned}$$

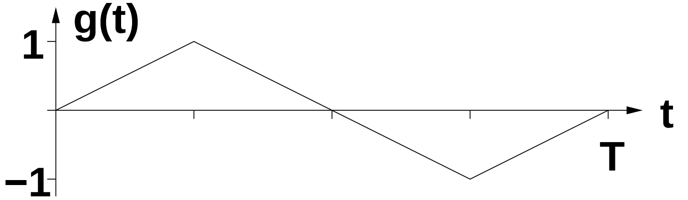
Which we can write out as follows $f(x) =$

$$\frac{8}{\pi^2}\left[-\sin\left(\frac{2\pi x}{L}\right)+\frac{\sin\left(\frac{2\pi 3x}{L}\right)}{9}-\frac{\sin\left(\frac{2\pi 5x}{L}\right)}{25}+\ldots\right]$$

Method 3: Maths Databook

Only works if something like the desired function is in the maths data book!





In this case we want $f(x)$ as above, and the nearest available series is $g(t)$.

$$g(t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin\left([2n-1]\omega_0 t\right)}{(2n-1)^2}$$

where $\omega_0 = 2\pi/T$.

If we set $x = t$ and $L = T$ then $f = -g$.

$$\begin{aligned} \Rightarrow f(x) &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left([2n-1]\omega_0 x\right)}{(2n-1)^2} (-1)^n \\ &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2\pi x(2n-1)}{L}\right)}{(2n-1)^2} (-1)^n \end{aligned}$$

Which we can write out, as with the other methods, as follows $f(x) =$

$$\frac{8}{\pi^2} \left[-\sin\left(\frac{2\pi x}{L}\right) + \frac{\sin\left(\frac{2\pi 3x}{L}\right)}{9} - \frac{\sin\left(\frac{2\pi 5x}{L}\right)}{25} + \ldots \right]$$

Section 6: Summary

$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{2\pi n x}{L}\right) f(x)\, dx$$

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\, dx$$

$$d = \frac{1}{L}\int_0^L f(x)\, dx$$

$$f(x) = d + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right]$$

You can sometimes combine multiple integrals using symmetry properties.

Sometimes it is faster to calculate a related Fourier series of delta functions and integrate.

Don't forget the Fourier serieses given in the maths data book.

Section 7

Convergence & Half Range Serieses

The rule for predicting the convergence of the Fourier series from the shape of the function is introduced.

This is used with the Fourier series for general period to calculate serieses, valid over limited ranges, with improved convergence properties. Four different serieses are calculated to model the same simple function in order to illustrate this.

The usefulness of Matlab and Octave for numerical calculation, and the use of Matlab for symbolic algebra are introduced.