Section 6: Summary

$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{2\pi n x}{L}\right) f(x)\,dx$$

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{2\pi n x}{L}\right) f(x)\,dx$$

$$d = \frac{1}{L}\int_0^L f(x)\,dx$$

$$f(x) = d + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right]$$

You can sometimes combine multiple integrals using symmetry properties.

Sometimes it is faster to calculate a related Fourier series of delta functions and integrate.

Don't forget the Fourier serieses given in the maths data book.

Section 7

Convergence & Half Range Serieses

The rule for predicting the convergence of the Fourier series from the shape of the function is introduced.

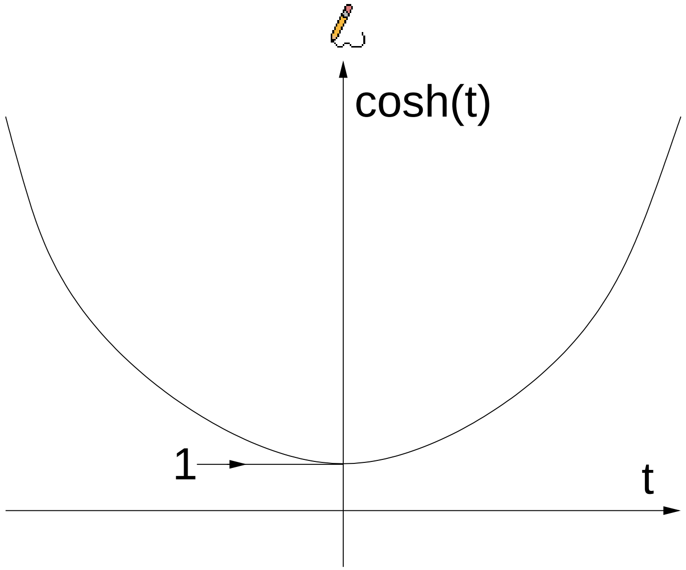
This is used with the Fourier series for general period to calculate serieses, valid over limited ranges, with improved convergence properties. Four different serieses are calculated to model the same simple function in order to illustrate this.

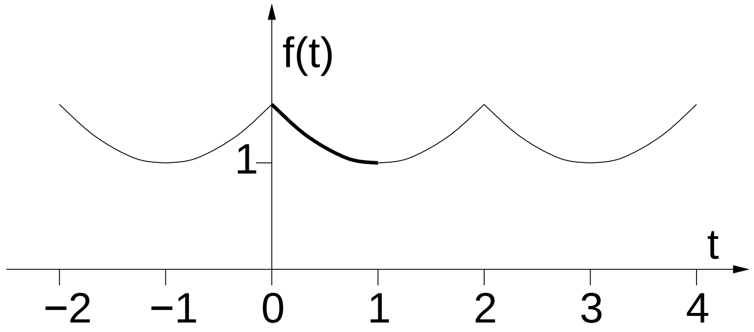
The usefulness of Matlab and Octave for numerical calculation, and the use of Matlab for symbolic algebra are introduced.

General Range Example 3

A even function $f(t)$ is periodic with period $T = 2$, and $f(t) = \cosh(t-1)$ for $0 \le t \le 1$. Sketch $f(t)$ in the range $-2 \le t \le 4$. Find a Fourier series representation for $f(t)$.

First remember what the graph of $\cosh(t)$ looks like.





It is an even function $\Rightarrow b_n = 0$ and $a_n \neq 0$. The mean value of the function is non-zero $\Rightarrow d \neq 0$.

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_{-1}^{1} f(t)\cos\left(\frac{2\pi n t}{T}\right)\,dt \\
&= \frac{4}{T}\int_{0}^{1} f(t)\cos\left(\frac{2\pi n t}{T}\right)\,dt \\
&= 2\int_{0}^{1} \cosh(t-1)\cos(n\pi t)\,dt \\
&= \frac{2\sinh(1)}{1+n^2\pi^2}
\end{aligned}
$$

$$d = \frac{1}{T}\int_{-1}^{1} f(t)\,dt = \frac{2}{T}\int_{0}^{1} f(t)\,dt$$

$$= \int_{0}^{1} \cosh(t-1)\,dt = \sinh(1)$$

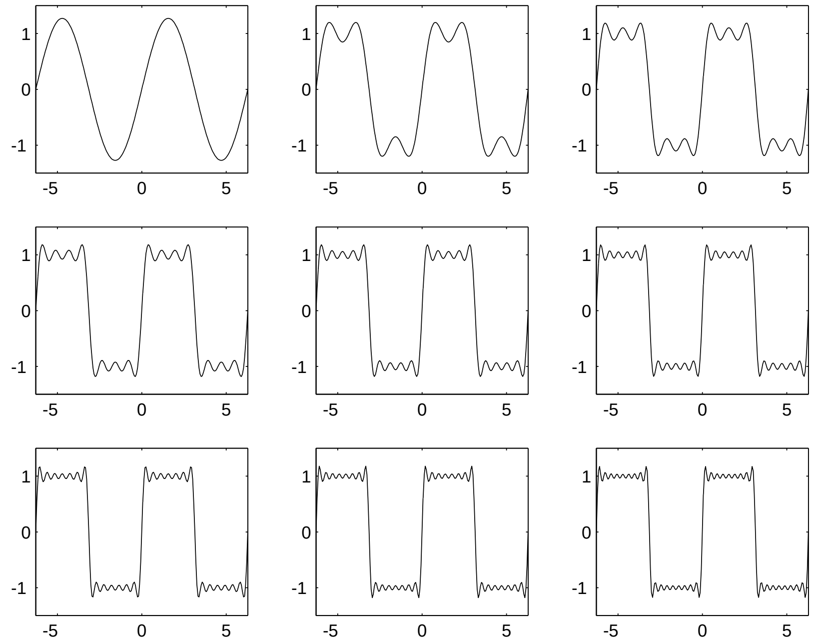
So

$$f(t) = d + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right)$$

$$= \sinh(1)\left[1 + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi t)}{1+n^2\pi^2}\right]$$

Square Wave Series Convergence

The graphs below show the sum of 1, 2, 3 . . . up to 9 terms of the Fourier series for a square wave.



Using Matlab/Octave

```
% Fourier series for square wave
number = 200;
dtheta = 4*pi/number;
theta = -2*pi:dtheta:2*pi;
nharm = 20;
d = 0;
thing = d * ones(1,number+1);
for n=1:nharm
  if mod(n,2) == 1
    bn = 4/(pi*n);
  else
    bn = 0;
  end
  an = 0;
  thing = thing + an * cos(n*theta) ...
                + bn * sin(n*theta);
  plot(theta,thing);
  axis([-2*pi 2*pi -1.5 1.5]);
  pause(1)
end
```



```
theta = -2*pi:dtheta:2*pi;
```

sets up theta as an array with 201 elements, starting at -2π, going up to 2π, with spacing `dtheta`$=4\pi/200$.

$-6.2832, -6.2204, \ldots \quad \ldots 6.2204, 6.2832$

```
thing = d * ones(1,number+1);
```

initialises the 201 element array in which we hold the value of the series at each angle. The initial value of each element is d, which in this case is zero.

```
for n=1:nharm
```

This introduces a `for` loop. We go round the loop `nharm` times to add in `nharm` harmonics.

```
thing = thing + an * cos(n*theta) ...
                + bn * sin(n*theta);
```

This statement works on every element of the `theta` array, calculating the terms of the cos and sinserieses and adding them in to the appropriate sums in the `thing` array.

Using Matlab Symbolic Tools

Both convolution and Fourier work involves a lot of integration. Sometimes it is nice to know what the right answer is, so you can check your working. To integrate p with respect to x from a to b you use the command `int(p, x, a, b)`. Consider the integral:

$$\frac{2}{T}\int_0^T \cos\left(\frac{2\pi n x}{T}\right) x\, dx = 0$$

```
>> syms n x T
>> int((2/T)*x*cos(2*pi*n*x/T),x,0,T)

ans =
T*(cos(pi*n)^2-1
 +2*pi*n*sin(pi*n)*cos(pi*n))/pi^2/n^2
```

which is

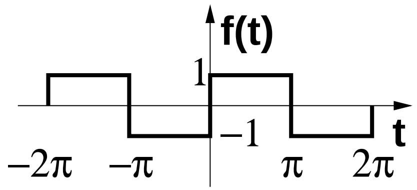
$$\frac{T\left((\cos(\pi\, n))^2 - 1 + 2\,\pi\, n \sin(\pi\, n)\cos(\pi\, n)\right)}{\pi^2 n^2}$$

But as n is an integer, $\cos^2(n\pi) = 1$ and $\sin(n\pi) = 0$, so the integral evaluates to zero.

Don't rely on this too much. You need to be able to integrate efficiently by hand in the exam.

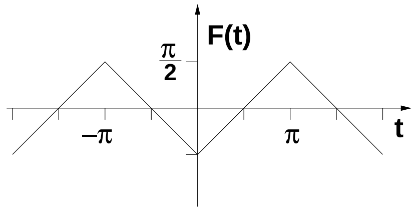
Convergence Examples

The Fourier series for a square wave converges as $1/n$. Notice that it is discontinuous of value.



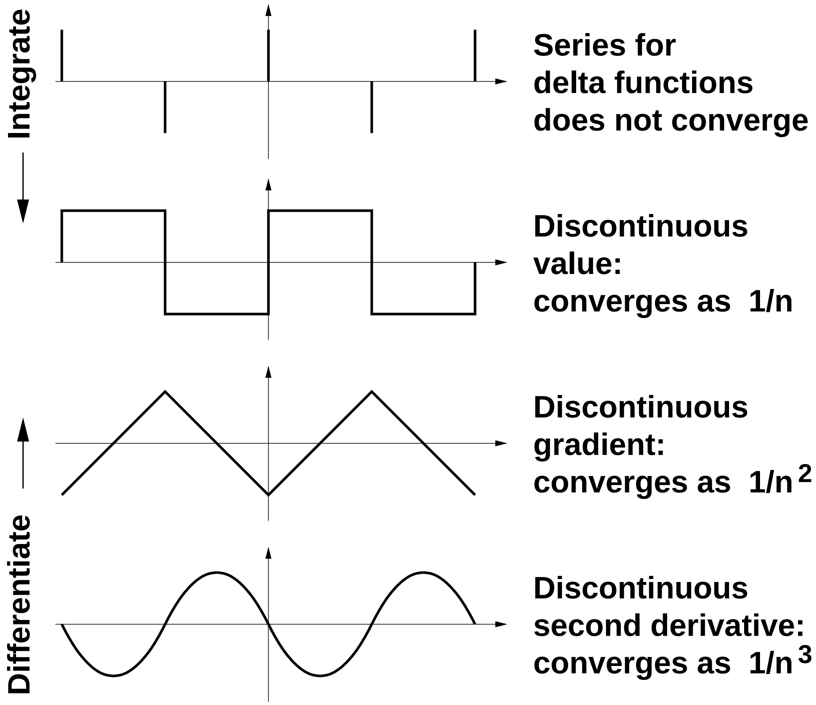
$$f(t) = \frac{4}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$$

The Fourier series for a triangular wave converges as $1/n^2$. It is continuous of value, but discontinuous of gradient.



$$F(t) = \frac{-4}{\pi}\left[\cos(t) + \frac{\cos(3t)}{9} + \frac{\cos(5t)}{25} + \ldots\right]$$

Convergence



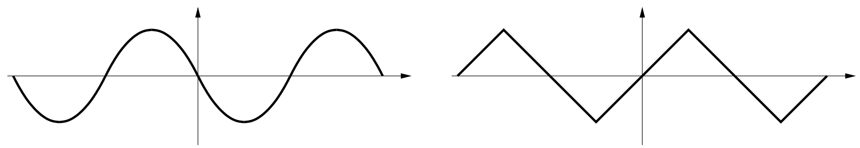
Odd Numbers

If $m = 1, 2, 3, 4, 5, 6, 7 \ldots$

and $n = 2m - 1$ and $m = \dfrac{n+1}{2}$

then $n = 1, 3, 5, 7, 9, 11, 13 \ldots$

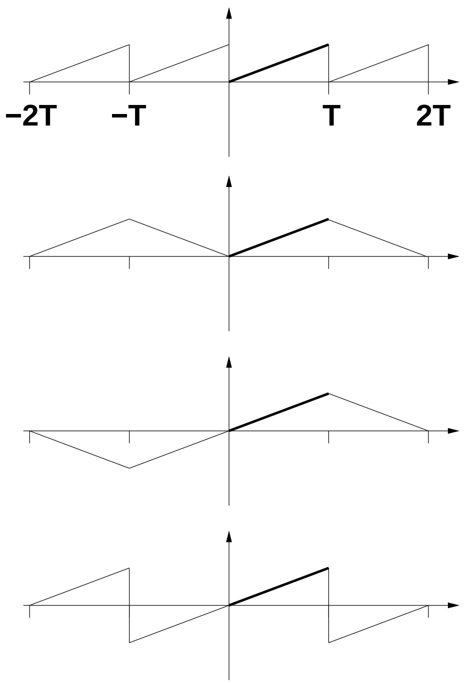
Odd Functions



$$f(x) = -f(-x)$$

“Half Range” Series

If we want to model a signal $f(x) = x$ in the range 0 to T. We can use the Fourier formulae for general range to generate a variety of different serieses. They will all be the same in the range 0 to T, but some may converge faster than others.



Full range series period T converges as 1/n

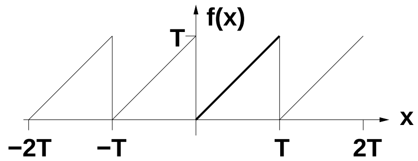
Cosine series period 2T, $b_n = 0$ converges as $1/n^2$

Sine series period 4T, $a_n = 0$, $d = 0$ converges as $1/n^2$

Sine series period 2T, $a_n = 0$, $d = 0$ converges as 1/n

Normal Series, Period T

Find the Fourier series to model $f(x) = x$ from 0 to T, using a series of period T.



$$a_n = \frac{2}{T}\int_0^T \cos\left(\frac{2\pi n x}{T}\right) x\,dx = 0$$

$$b_n = \frac{2}{T}\int_0^T \sin\left(\frac{2\pi n x}{T}\right) x\,dx = \frac{-T}{n\pi}$$

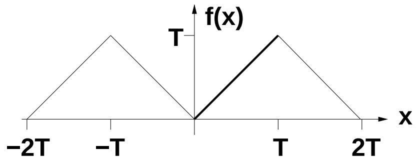
$$d = \frac{1}{T}\int_0^T x\,dx = \frac{T}{2}$$

$$\Rightarrow f(x) = \frac{T}{2} - \sum_{n=1}^{\infty}\frac{T}{n\pi}\sin\left(\frac{2\pi n x}{T}\right)$$

Notice that the series converges as $1/n$.

Cosine Series, Period 2T

Find the Fourier series to model $f(x) = x$ from 0 to T, using a cosine series of period $2T$.



$$a_n = \frac{1}{T}\int_{-T}^{T} \cos\left(\frac{\pi n x}{T}\right) f(x)\, dx$$

$$= \frac{1}{T}\left[\int_{-T}^{0} \cos\left(\frac{\pi n x}{T}\right)(-x)\, dx + \int_{0}^{T} \cos\left(\frac{\pi n x}{T}\right) x\, dx\right]$$

$$= \frac{2}{T}\int_{0}^{T} \cos\left(\frac{\pi n x}{T}\right) x\, dx = \frac{-4T}{n^2\pi^2}, \text{ only } n \text{ ODD}$$

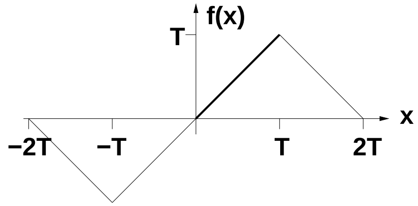
$$d = \frac{1}{2T}\int_{-T}^{T} f(x)\, dx = \frac{T}{2}$$

$$f(x) = \frac{T}{2} - \sum_{m=1}^{\infty} \frac{4T}{(2m-1)^2\pi^2} \cos\left(\frac{(2m-1)\pi x}{T}\right)$$

Notice that the series converges as $1/n^2$.

Sine Series, Period 4T

Find the Fourier series to model $f(x) = x$ from 0 to T, using a sine series of period $4T$.



Notice how the function is symmetrical about T (i.e. $\frac{1}{4}$ of the period). This leads to $b_n = 0$ when n is even because all such terms are anti-symmetric about T.

$$b_n = \frac{1}{2T}\int_{-2T}^{2T} \sin\left(\frac{\pi n x}{2T}\right) f(x)\, dx$$

$$= \frac{1}{T}\int_{-T}^{T} \sin\left(\frac{\pi n x}{2T}\right) x\, dx, \; n \text{ odd only}$$

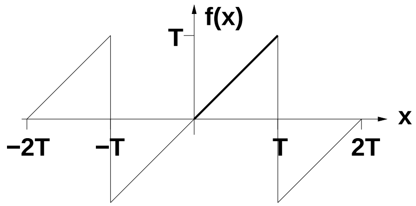
$$= \frac{8T(-1)^{\left(\frac{n+3}{2}\right)}}{n^2\pi^2}, \; n \text{ odd only}$$

$$f(x) = \sum_{m=1}^{\infty} \frac{8T(-1)^{(m+1)}}{(2m-1)^2\pi^2} \sin\left(\frac{\pi(2m-1)x}{2T}\right)$$

Notice that the series converges as $1/n^2$.

Sine Series, Period 2T

Find the Fourier series to model $f(x) = x$ from 0 to T, using a sine series of period $2T$.



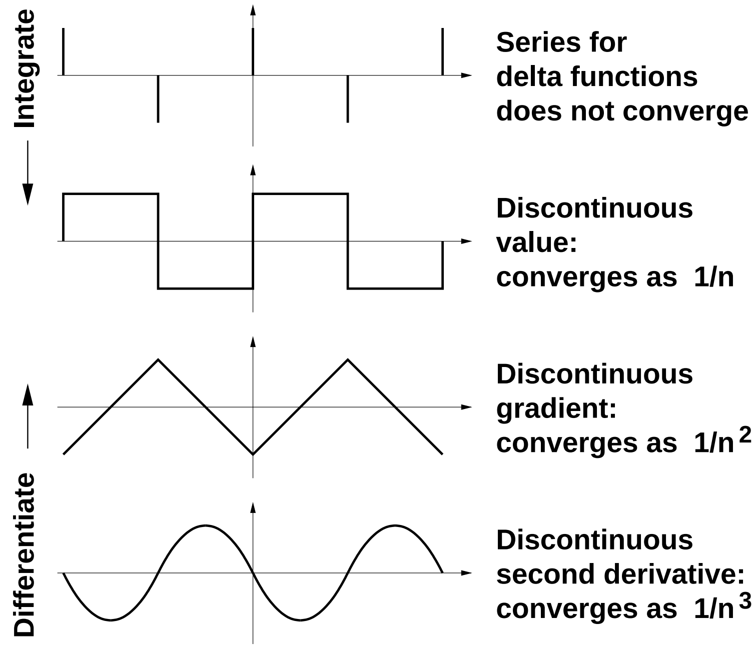
$$b_n = \frac{1}{T}\int_{-T}^{T} \sin\left(\frac{\pi n x}{T}\right) x\,dx$$

$$= \frac{-2T}{n\pi}(-1)^n$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{-2T}{n\pi}(-1)^n \sin\left(\frac{\pi n x}{T}\right)$$

Notice that the series converges as $1/n$.

Section 7: Summary



If you are modelling a limited section of a function, pick the Fourier series period so as to get good convergence and a series that is easy to calculate (i.e. some of a_n, b_n or d zero).