Section 8

Complex Fourier Series

The complex Fourier series is presented first with period 2π, then with general period.

The connection with the real-valued Fourier series is explained and formulae are given for converting between the two types of representation.

Examples are given of computing the complex Fourier series and converting between complex and real serieses.

New Basis Functions

Recall that the Fourier series builds a representation composed of a weighted sum of the following basis functions.

1 (i.e. a constant term)

$\cos(t)$	$\cos(2t)$	$\cos(3t)$	$\cos(4t)$	$\ldots$
$\sin(t)$	$\sin(2t)$	$\sin(3t)$	$\sin(4t)$	$\ldots$

Computing the weights a_n, b_n and c often involves some nasty integration.

We now present an alternative representation based on a different set of basis functions:

1 (i.e. a constant term)

e^{it}	e^{2it}	e^{3it}	e^{4it}	$\ldots$
e^{-it}	e^{-2it}	e^{-3it}	e^{-4it}	$\ldots$

These can all be represented by the term

$$e^{int}$$

with n taking integer values from $-\infty$ to $+\infty$. Note that the constant term is provided by the case when $n = 0$.

Series of Complex Exponentials

A representation based on this family of functions is called the "complex Fourier series".

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

The coefficients, c_n, are normally complex numbers.

It is often easier to calculate than the sin/cos Fourier series because integrals with exponentials in are usually easy to evaluate.

We will now derive the complex Fourier series equations, as shown above, from the sin/cos Fourier series using the expressions for sin() and cos() in terms of complex exponentials.

Complex Fourier Series

$$f(t) = d + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)]$$

$$= d + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{int} + e^{-int}}{2} \right) + b_n \left(\frac{e^{int} - e^{-int}}{2i} \right) \right]$$

$$= d + \sum_{n=1}^{\infty} \frac{(a_n - ib_n)}{2} e^{int} + \sum_{n=1}^{\infty} \frac{(a_n + ib_n)}{2} e^{-int}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{int}$$

where

$$c_n = \begin{cases} d & , n = 0 \\ (a_n - ib_n)/2 & , n = 1, 2, 3, \ldots \\ (a_{-n} + ib_{-n})/2 & , n = -1, -2, -3, \ldots \end{cases}$$

Note that a_{-n} and b_{-n} are only defined when n is negative.

$$
\begin{aligned}
a_n &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} \cos(nt) f(t)\, dt \\
b_n &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} \sin(nt) f(t)\, dt \\
d &= \tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi} f(t)\, dt
\end{aligned}
$$

thus for n positive

$$
\begin{aligned}
c_n &= \frac{1}{2}(a_n - ib_n) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos(nt) - i\sin(nt)]\, f(t)\, dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\, dt
\end{aligned}
$$

for n negative

$$
\begin{aligned}
c_n &= \frac{1}{2}(a_{-n} + ib_{-n}) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos(-nt) + i\sin(-nt)]\, f(t)\, dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\, dt
\end{aligned}
$$

and for $n = 0$

$$
\begin{aligned}
c_0 &= d \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-0} f(t)\, dt
\end{aligned}
$$

Complex Fourier Series Summary

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\, dt \\
f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{int}
\end{aligned}
$$

Complex Series Example 1

Find the complex Fourier series to model $f(t) = \sin(t)$.

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\sin(t)\,dt \\ &= \frac{1}{2\pi}\left[\frac{e^{in\pi} - e^{-in\pi}}{n^2 - 1}\right] \end{aligned}$$

Which is zero when n does not equal 1 or -1. For these two special cases we have to set $n = 1 + \epsilon$ and calculate the limit of c_n as ϵ tends to zero. This gives us

$$\begin{aligned} c_1 &= \frac{1}{2i} \\ c_{-1} &= \frac{-1}{2i} \end{aligned}$$

Which means the complex Fourier series for $f(t) = \sin(t)$ is

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{int} \\ &= \frac{e^{it} - e^{-it}}{2i} \end{aligned}$$

Finding the limit as n tends to 1

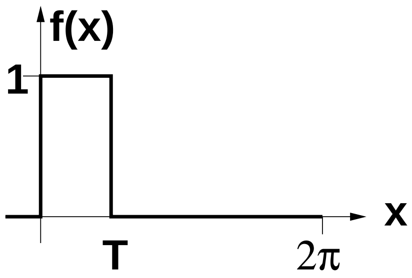
$$c_n = \frac{1}{2\pi}\left[\frac{e^{in\pi} - e^{-in\pi}}{n^2 - 1}\right]$$

Set $n = 1 + \epsilon$ and let ϵ tend to zero.

$$\begin{aligned} c_1 &= \frac{1}{2\pi}\left[\frac{e^{i\pi(1+\epsilon)} - e^{-i\pi(1+\epsilon)}}{(1+\epsilon)^2 - 1}\right] \\ &= \frac{1}{2\pi}\left[\frac{-e^{i\pi\epsilon} + e^{-i\pi\epsilon}}{(1+\epsilon)^2 - 1}\right] \\ &\approx \frac{1}{2\pi}\left[\frac{-1 - i\pi\epsilon + 1 - i\pi\epsilon}{1 + 2\epsilon - 1}\right] \\ &\approx \frac{1}{2\pi}\left[\frac{-2i\pi\epsilon}{2\epsilon}\right] \\ &\approx \frac{-i}{2} \\ &\approx \frac{1}{2i} \end{aligned}$$

Complex Series Example 2

Find the complex Fourier series to model $f(x)$ that has a period of 2π and is 1 when $0 < x < T$ and zero when $T < x < 2\pi$.



$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt \\ &= \frac{i}{2\pi n}\left[e^{-inT} - 1\right] \text{, when } n \neq 0 \\ &= \frac{1}{2\pi}\text{area} = \frac{T}{2\pi} \text{, when } n = 0 \end{aligned}$$

So the Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

$$= \frac{1}{2\pi}\left\{T + \sum_{n=-\infty}^{-1} \frac{i}{n}\left[e^{-inT} - 1\right] e^{int} + \sum_{n=1}^{\infty} \frac{i}{n}\left[e^{-inT} - 1\right] e^{int}\right\}$$

Converting c to a, b and d

From our example on the previous page.

$$c_n = \begin{cases} \frac{i}{2\pi n}\left[e^{-inT} - 1\right] & \text{, when } n \neq 0 \\ \frac{1}{2\pi}\text{area} = \frac{T}{2\pi} & \text{, when } n = 0 \end{cases}$$

We wish to calculate the coefficients for the equivalent Fourier series in terms of $\sin()$ and $\cos()$.

Clearly $d = c_0 = \frac{T}{2\pi}$. For $n > 0$

$$\begin{aligned} c_n &= (a_n - ib_n)/2 \\ \Rightarrow a_n &= 2\,\mathbf{Re}\{c_n\} \\ \text{and } b_n &= -2\,\mathbf{Im}\{c_n\} \end{aligned}$$

converting our expression for c_n into sin() and cos():

$$\begin{aligned} 2c_n &= \frac{i}{\pi n}\left[\cos(nT) - i\sin(nT) - 1\right] \\ &= \frac{1}{\pi n}\left[\sin(nT) + i(\cos(nT) - 1)\right] \end{aligned}$$

so $a_n = \dfrac{\sin(nT)}{n\pi}$ and $b_n = \dfrac{1 - \cos(nT)}{n\pi}$.

Complex Fourier Series

$$f(t) = \frac{1}{2\pi}\left\{T + \sum_{n=-\infty}^{-1} \frac{i}{n}\left[e^{-inT} - 1\right] e^{int} + \sum_{n=1}^{\infty} \frac{i}{n}\left[e^{-inT} - 1\right] e^{int}\right\}$$

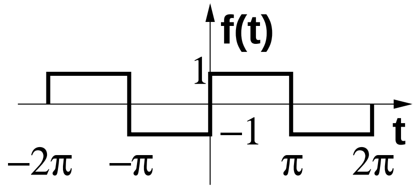
Real Fourier Series

$$f(t) = \frac{T}{2\pi} + \sum_{n=1}^{\infty} \frac{\sin(nT)}{n\pi}\cos(nt) + \sum_{n=1}^{\infty} \frac{1-\cos(nT)}{n\pi}\sin(nt)$$

Both serieses converge as $1/n$.

Converting from Real to Complex

Convert the real Fourier series of the square wave $f(t)$ to a complex series.



For the real series, we know that $d = a_n = 0$ and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt) f(t)\, dt = \frac{4}{n\pi}, \; n \text{ odd}$$

giving $f(t) = \frac{4}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \ldots\right]$

To convert to a complex series, use

$$c_n = \begin{cases} d & , n = 0 \\ (a_n - ib_n)/2 & , n = 1, 2, 3, \ldots \\ (a_{-n} + ib_{-n})/2 & , n = -1, -2, -3, \ldots \end{cases}$$

so we have

$$\begin{aligned} c_0 &= 0 \\ c_n &= -2i/(n\pi) \quad , n \text{ positive and odd} \\ c_n &= 2i/(-n\pi) \quad , n \text{ negative and } |n| \text{ odd} \end{aligned}$$

$$\Rightarrow f(t) = \frac{-2i}{\pi}\left[\ldots + \frac{e^{-5it}}{-5} + \frac{e^{-3it}}{-3} + \frac{e^{-it}}{-1} + \frac{e^{it}}{1} + \frac{e^{3it}}{3} + \frac{e^{5it}}{5} + \ldots\right]$$

General Complex Series

For period of 2π

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)\, dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

Similarly, for period L

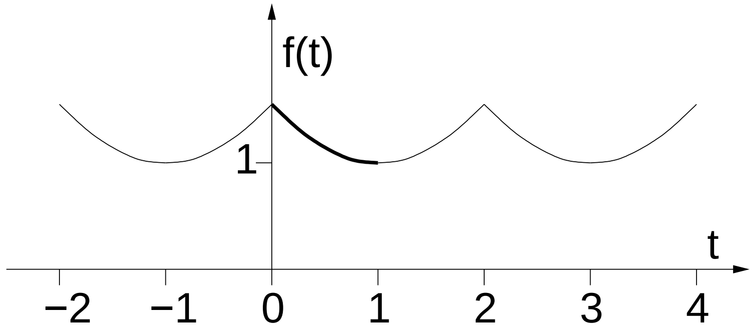
$$c_n = \frac{1}{L}\int_0^{L} e^{-inx\frac{2\pi}{L}} f(x)\, dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx\frac{2\pi}{L}}$$

The fraction $\frac{2\pi}{L}$ is often written as ω_0 and called the fundamental angular frequency.

Example 1

A even function $f(t)$ is periodic with period $L = 2$, and $f(t) = \cosh(t-1)$ for $0 \le t \le 1$. Find a complex Fourier series representation for $f(t)$.



$$\begin{aligned} c_n &= \frac{1}{L}\int_0^{L} e^{-int\frac{2\pi}{L}} f(t)\, dt \\ &= \frac{1}{2}\int_0^{2} e^{-int\pi} \cosh(t-1)\, dt \\ &= \frac{\sinh(1)}{1+n^2\pi^2} \end{aligned}$$

Hence the complex Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int\frac{2\pi}{L}}$$
$$= \sum_{n=-\infty}^{\infty} \frac{\sinh(1) e^{int\pi}}{1+n^2\pi^2}$$

We can check this answer by computing the equivalent real Fourier series which we calculated at the start of section 7.

$$\begin{aligned} a_n &= 2\,\mathbf{Re}\{c_n\} \quad , n = 1, 2, 3, \ldots \\ b_n &= -2\,\mathbf{Im}\{c_n\} \quad , n = 1, 2, 3, \ldots \\ d &= c_0 \end{aligned}$$

In this case, as c_n is entirely real,

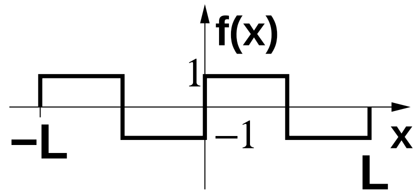
$$a_n = 2c_n = \frac{2\sinh(1)}{1+n^2\pi^2} \; , n = 1, 2, 3, \ldots$$

$$b_n = 0$$

$$d = \sinh(1)$$

Example 2

Find the complex Fourier series of the the square wave $f(x)$.



Note that the mean of the function is zero, so $c_0 = 0$.

$$\begin{aligned} c_n &= \frac{1}{L}\int_0^L e^{-inx\frac{2\pi}{L}} f(x)\,dx \\ &= \frac{1}{L}\left[\int_0^{L/2} e^{-inx\frac{2\pi}{L}}\,dx - \int_{L/2}^{L} e^{-inx\frac{2\pi}{L}}\,dx\right] \\ &= \frac{1}{2in\pi}\left[e^{-2in\pi} + 1 - 2e^{-in\pi}\right] \end{aligned}$$

$$f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\left[1 - e^{-in\pi}\right]}{in\pi} e^{inx\frac{2\pi}{L}}$$

$$f(x) = \frac{2}{i\pi}\left[\ldots + \frac{e^{-5ix\frac{2\pi}{L}}}{-5} + \frac{e^{-3ix\frac{2\pi}{L}}}{-3} + \frac{e^{-ix\frac{2\pi}{L}}}{-1} + \frac{e^{ix\frac{2\pi}{L}}}{1} + \frac{e^{3ix\frac{2\pi}{L}}}{3} + \frac{e^{5ix\frac{2\pi}{L}}}{5} + \ldots\right]$$

Converting to a Real Series

We wish to convert the complex general range square wave series into a series with real coefficients.

$$c_n = \begin{cases} 2/(in\pi) & , |n| \text{ odd} \\ 0 & , |n| \text{ even} \end{cases}$$

Clearly $d = c_0 = 0$. For a and b use:

$$\begin{aligned} c_n &= (a_n - ib_n)/2 \\ \Rightarrow a_n &= 2\,\textbf{Re}\{c_n\} = 0 \\ \text{and } b_n &= -2\,\textbf{Im}\{c_n\} = \frac{4}{n\pi},\ n \text{ odd} \end{aligned}$$

Which gives us the real series:

$$f(t) = \frac{4}{\pi}\left[\sin\left(x\frac{2\pi}{L}\right) + \frac{\sin\left(3x\frac{2\pi}{L}\right)}{3} + \frac{\sin\left(5x\frac{2\pi}{L}\right)}{5} + \dots\right]$$

Section 8: Summary

For period L

$$\begin{aligned} c_n &= \frac{1}{L}\int_0^L e^{-inx\frac{2\pi}{L}} f(x)\,dx \\ f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx\frac{2\pi}{L}} \end{aligned}$$

Relationship with the cos/sin Fourier series.

$$c_n = \begin{cases} d & , n = 0 \\ (a_n - ib_n)/2 & , n = 1, 2, 3, \dots \\ (a_{-n} + ib_{-n})/2 & , n = -1, -2, -3, \dots \end{cases}$$

$$\begin{aligned} a_n &= 2\,\textbf{Re}\{c_n\} & , n = 1, 2, 3, \dots \\ b_n &= -2\,\textbf{Im}\{c_n\} & , n = 1, 2, 3, \dots \\ d &= c_0 \end{aligned}$$