Section 9

Probability

In this section we summarise the key issues in the basic probability teach-yourself document and provide a single simple example of each concept.

This presentation is intended to be reinforced by the many examples in the teach-yourself document and examples paper 10.

Probability

Probability of A $=$

$$\frac{\text{Number of outcomes for which } A \text{ happens}}{\text{Total number of outcomes (sample space)}}$$

What is the probability of drawing an ace from a shuffled pack of cards?

There are 4 aces. There are 52 cards in total. Therefore the probability is

$$\mathbf{P}(\text{ ace }) = \frac{4}{52} = \frac{1}{13}$$

Adding Probabilities

$$\mathbf{P}(A \text{ or } B) = \mathbf{P}(A) + \mathbf{P}(B)$$

provided A and B cannot happen together, i.e. A and B must be mutually *exclusive* outcomes.

What is the probability of drawing an ace or a king from a shuffled pack of cards?

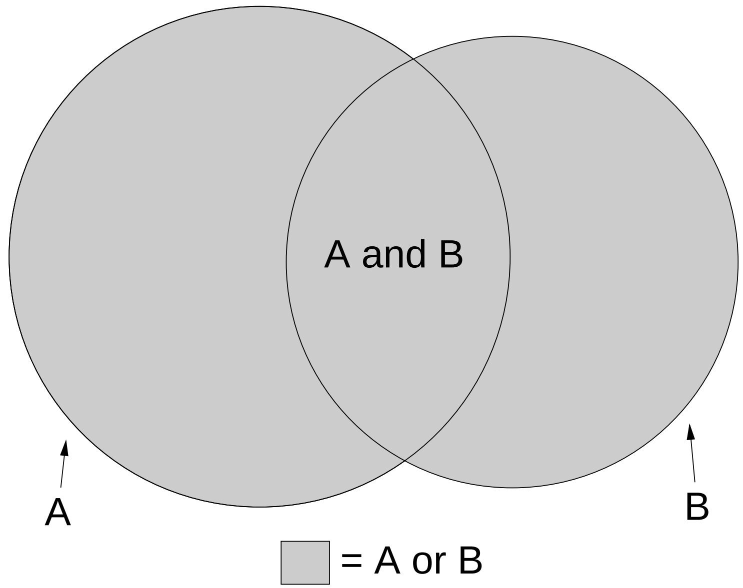
$$\mathbf{P}(\text{ ace }) = \frac{1}{13}$$

$$\mathbf{P}(\text{ king }) = \frac{1}{13}$$

$$\Rightarrow \mathbf{P}(\text{ ace or king }) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

When Not to Add Probabilities

When the events are not mutually exclusive.



$$\mathbf{P}(A \text{ or } B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \text{ and } B)$$

Non-Exclusive Events

What is the probability of drawing an ace or a spade from a shuffled pack of cards?

$$\mathbf{P}(\text{ ace }) = \frac{1}{13} \qquad \mathbf{P}(\text{ spade }) = \frac{13}{52} = \frac{1}{4}$$

but **P**(ace or spade) is not the sum of these values because the outcomes “ace” and “spade” are not exclusive; it is possible to have them both together by drawing the ace of spades.

To calculate **P**(ace or spade)

either use the formula from the previous slide:

$$\begin{array}{ccccccc} \mathbf{P}(\text{ ace }) & + & \mathbf{P}(\text{ spade }) & - & \mathbf{P}(\text{ ace of spades }) & & \\ \frac{1}{13} & + & \frac{1}{4} & - & \frac{1}{52} & = & \frac{4}{13} \end{array}$$

or use the original definition of probability.

$$\frac{\text{number of aces and spades}}{\text{total number of cards}} = \frac{4+13-1}{52} = \frac{4}{13}$$

Multiplying Probabilities

$$\mathbf{P}(A \text{ and } B) = \mathbf{P}(A) \times \mathbf{P}(B)$$

provided A is not affected by the outcome of B and B is not affected by the outcome of A, i.e. A and B must be *independent*.

I have two shuffled packs of cards and draw a card from each of them. What is the probability that I draw two aces?

$$\mathbf{P}(\text{ ace }) = \frac{1}{13}$$

$$\mathbf{P}(\text{ ace and ace }) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$$

Non-independent Events

I have a single pack of cards. I draw a card, then draw a second card without putting the first card back in the pack. What is the probability that I draw two aces?

This time the probability that I get an ace as the second card is affected by whether or not I removed an ace from the pack when I drew the first card.

We use the notation $\mathbf{P}(B|A)$ to denote the probability that B happens, given that we know that A happened. This is called a conditional probability.

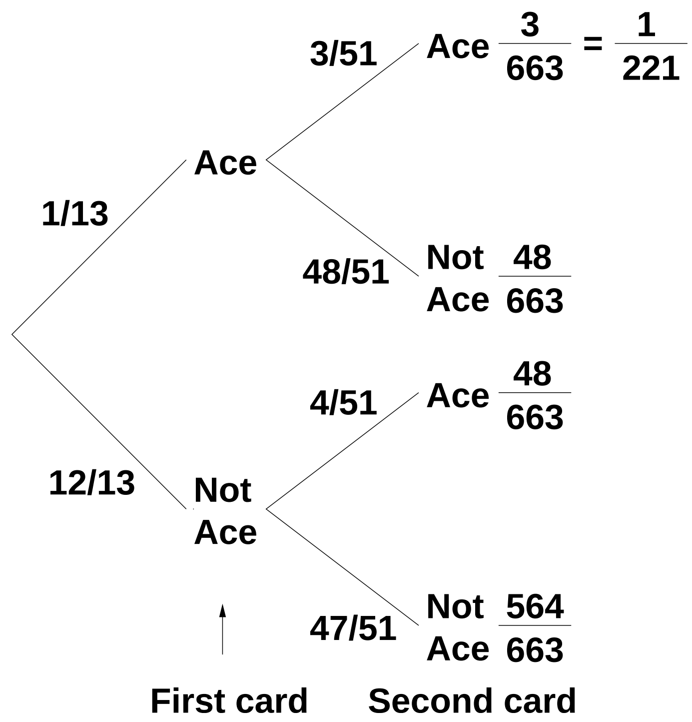
$$\mathbf{P}(A \text{ and } B) = \mathbf{P}(A|B)\,\mathbf{P}(B)$$

Thus:

$$\begin{aligned}&\mathbf{P}(\text{ [second = ace] and [first = ace] })\\ &\quad = \mathbf{P}(\text{ second = ace } | \text{ first = ace })\,\mathbf{P}(\text{ first = ace })\end{aligned}$$

Tree Diagram

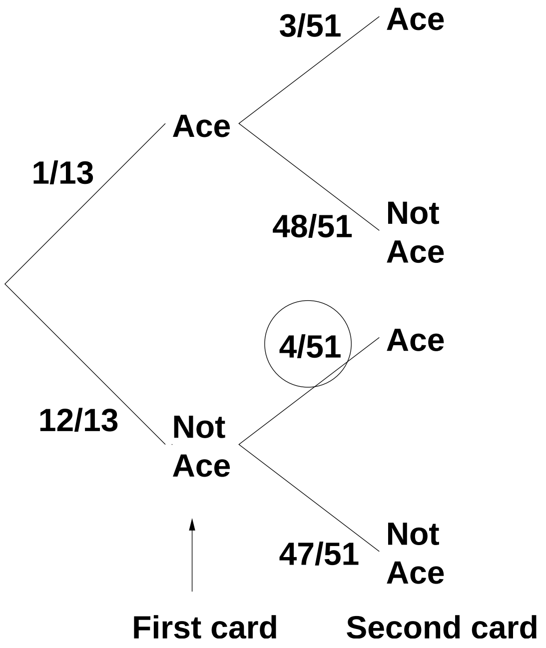
We can use a tree diagram to help us work this out.



The probability that both cards are aces $= \frac{1}{221}$.

Another Example

I have a single pack of cards. I draw a card, then draw a second card without putting the first card back in the pack. What is the probability that the second card is an ace, given that the first card was not an ace?



$\mathbf{P}(\text{ second = ace} \mid \text{first = not ace }) = \frac{4}{51}$

Notation

Intersection	$\bigcap$	AND
Union	$\bigcup$	OR

Thus the conditional probability formula

$$\mathbf{P}(A \text{ and } B) = \mathbf{P}(A|B)\,\mathbf{P}(B)$$

is more normally written

$$\mathbf{P}(A \cap B) = \mathbf{P}(A|B)\,\mathbf{P}(B)$$

and instead of

$$\mathbf{P}(A \text{ or } B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \text{ and } B)$$

we write

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$

Ordering Objects

The number of different orders in which n unique objects can be placed is $n!$ (n factorial)

I have three cards with values 2, 3 and 4. They are shuffled into a random order. What is the probability they are in the order 2, 3, 4?

The number of possible orders for three cards is $3!$ The probability the cards are found in one specific order is therefore $\frac{1}{3!} = \frac{1}{6}$.

Permutations

$$_nP_r = \frac{n!}{(n-r)!}$$

is the number of ways of choosing r items from n when the order of the chosen items matters.

Ten people are involved in a race. I wish to make a poster for every possible winning combination of gold, silver and bronze medal winners. How many posters will I need?

We need to know the number of ways of choosing three people out of 10, taking account of the order. This is

$$_{10}P_3 = \frac{10!}{7!} = 10 \times 9 \times 8 = 720$$

So I would need rather a lot of posters.

Combinations

$$ {}_{n}C_{r} = \frac{n!}{(n-r)!r!} $$

is the number of ways of choosing r items from n when the order of the chosen items does not matter.

I have a single pack of cards. I draw a card, then draw a second card without putting the first card back in the pack. What is the probability that I draw two aces?

The number of ways of drawing 2 cards from 52 is ${}_{52}C_{2}$.

The number of ways of getting two aces is the number of ways of drawing 2 aces from the 4 aces in the pack. This is ${}_{4}C_{2}$.

The probability that I draw two aces is therefore

$$ \frac{\text{num ace pairs}}{\text{num pairs}} = \frac{{}_{4}C_{2}}{{}_{52}C_{2}} = \frac{4!}{2!\,2!} \times \frac{50!\,2!}{52!} = \frac{4 \times 3}{52 \times 51} = \frac{1}{221} $$

Lottery Example 1

If what is the probability of winning the jackpot in the national lottery? There are 49 balls and you have to match all six to win.

Method 1:

$$ \frac{6}{49} \times \frac{5}{48} \times \frac{4}{47} \times \frac{3}{46} \times \frac{2}{45} \times \frac{1}{44} = \frac{1}{13983816} $$

Method 2:

$$ \frac{1}{\text{number of ways of choosing 6 balls from 49, where order does not matter}} = \frac{1}{{}_{49}C_{6}} $$

$$ \frac{1}{{}_{49}C_{6}} = \frac{6!\,43!}{49!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{49 \times 48 \times 47 \times 46 \times 45 \times 44} = \frac{1}{13983816} $$

Lottery Example 2

What is the probability of winning £10 by matching exactly 3 balls in the national lottery

Method 1:

Work out the probability of matching them in a particular order: the first 3 balls that are drawn win, the remaining 3 do not.

$$\frac{6}{49} \times \frac{5}{48} \times \frac{4}{47} \times \frac{43}{46} \times \frac{42}{45} \times \frac{41}{44}$$

Then multiply this by the number of possible ways of picking the 3 winning balls among the 6 balls that are drawn.

$$\left.\begin{array}{cccccc} ✔ & ✔ & ✔ & ✘ & ✘ & ✘ \\ ✔ & ✔ & ✘ & ✔ & ✘ & ✘ \\ ✔ & ✔ & ✘ & ✘ & ✔ & ✘ \\ ✔ & ✔ & ✘ & ✘ & ✘ & ✔ \\ ✔ & ✘ & ✔ & ✔ & ✘ & ✘ \\ \ldots \text{etc.} & & & & & \end{array}\right\} {}_6C_3 = 20$$

Hence: $\frac{6}{49} \times \frac{5}{48} \times \frac{4}{47} \times \frac{43}{46} \times \frac{42}{45} \times \frac{41}{44} \times {}_6C_3 = \frac{1}{56.7}$

Matching only 3 balls, method 2:

$$\frac{\text{number of ways we can win}}{\text{total possible number of outcomes}}$$

Thinking about all the balls in the lottery machine, we consider:

$$\frac{\left(\begin{array}{l}\text{The number of ways the lottery machine can pick}\\ \text{3 balls matching some of the 6 numbers on our ticket.}\end{array}\right)\left(\begin{array}{l}\text{The number of ways the lottery machine can pick}\\ \text{3 balls from the 43 balls not on our ticket.}\end{array}\right)}{\left(\begin{array}{l}\text{The total number of ways of picking 6}\\ \text{balls out of the 49 in the machine.}\end{array}\right)}$$

$$= \frac{{}_6C_3\, {}_{43}C_3}{{}_{49}C_6} = \frac{6!\,43!\,43!\,6!}{49!\,40!\,3!\,3!\,3!}$$

$$= \frac{43 \times 42 \times 41 \times 6 \times 5 \times 4 \times 6 \times 5 \times 4}{49 \times 48 \times 47 \times 46 \times 45 \times 44 \times 3 \times 2}$$

$$= \frac{1}{56.7}$$

Section 9: Summary

$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$ if A and B are mutually exclusive outcomes.

$\mathbf{P}(A \cap B) = \mathbf{P}(A) \times \mathbf{P}(B)$ provided A and B are independent.

$\mathbf{P}(A \cap B) = \mathbf{P}(A|B)\,\mathbf{P}(B)$

$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$

The number of different orders in which n unique objects can be placed is $n!$

Permutations: ${}_nP_r = \frac{n!}{(n-r)!}$ is the number of ways of choosing r items from n when the order of the chosen items matters.

Combinations: ${}_nC_r = \frac{n!}{(n-r)!r!}$ is the number of ways of choosing r items from n when the order of the chosen items does not matter.

Acknowledgements

These notes are inspired by Prof Woodhouse's notes for the same course and incorporate several suggestions from Drs Gee and Treece. I am very grateful to Martin Weber for checking the algebra in the examples.